

FULL RANGE OF BLOW UP EXPONENTS FOR THE QUINTIC WAVE EQUATION IN THREE DIMENSIONS

JOACHIM KRIEGER, WILHELM SCHLAG

ABSTRACT. For the critical focusing wave equation $\square u = u^5$ on \mathbb{R}^{3+1} in the radial case, we prove the existence of type II blow up solutions with scaling parameter $\lambda(t) = t^{-1-\nu}$ for all $\nu > 0$. This extends the previous work by the authors and Tataru where the condition $\nu > \frac{1}{2}$ had been imposed, and gives the optimal range of polynomial blow up rates in light of recent work by Duyckaerts, Kenig and Merle.

1. INTRODUCTION

We consider the energy critical focussing wave equation

$$\square u = u^5, \square = \partial_t^2 - \Delta \quad (1.1)$$

on \mathbb{R}^{3+1} , in the radial case. This equation has been intensely studied in a number of recent works: the remarkable series of papers [3] - [6] established a complete classification of all *possible* type II blow up dynamics, without proving their existence. In the works [8], [2] a constructive approach to actually exhibit and thereby prove the existence of such type II dynamics was undertaken. Recall that a type II blow up solution $u(t, x)$ with blow up time T_* is one for which

$$\limsup_{t \rightarrow T_*} (\|u(t, \cdot)\|_{\dot{H}^1} + \|u_t(t, \cdot)\|_{L_x^2}) < \infty$$

In [6], it is demonstrated that such solutions can be described as a sum of dynamically re-scaled ground states

$$\pm W(x) = \pm \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$$

plus a radiation term. In particular, for solutions where only one such bulk term is present, one can write the solution as

$$u(t, x) = W_{\lambda(t)}(x) + w(t, x) + o_{\dot{H}^1}(1), \quad W_\lambda = \lambda^{\frac{1}{2}} W(\lambda x), \quad w(t, x) \in \dot{H}^1 \quad (1.2)$$

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where the “error” $(w(t, \cdot), \partial_t w(t, \cdot))$ converges strongly in $\dot{H}^1 \times L^2$ as $t \rightarrow T_*$, and we have the dynamic condition

$$\lim_{t \rightarrow T_*} (T_* - t) \lambda(t) = \infty \quad (1.3)$$

In [8], it was shown that such solutions with $\lambda(t) = t^{-1-\nu}$ do exist, where $\nu > \frac{1}{2}$ is arbitrary. This left the question whether for *polynomial* rates the condition (1.3) is indeed optimal. Here we show that it is.

Theorem 1.1. *Let $\nu > 0$ be given. Then there exists an energy class solution $u(t, x)$, which in fact has regularity $H^{1+\frac{\nu}{2}-}$, of the form (1.2), with*

$$\lambda(t) = t^{-1-\nu}$$

Our method of proof is closely modeled on the construction from [8], of which we now recall the main steps:

- (i) We write $u_0(t, r) = W_{\lambda(t)}(r)$ and iteratively modify u_0 in the form

$$u_{2k-1} = u_0 + \sum_{j=1}^{2k-1} v_j$$

so that u_{2k-1} satisfies (1.1) up to an error of size t^N as $t \rightarrow 0+$; here N can be made as large as desired by taking k large, and the size is measured relative to the energy inside a light cone with tip at $r = 0, t = 0$.

- (ii) We seek an exact solution via a perturbation: $u = u_{2k-1} + \varepsilon$. To solve for ε we switch to coordinates $R = \lambda(t)r, \tau = \int_t^\infty \lambda(s) ds = \frac{1}{\nu} t^{-\nu}$. The variable τ varies in the range $\tau_0 \leq \tau < \infty$.
- (iii) In the new coordinates, the driving linear operator is

$$\mathcal{L} = -\partial_{RR} - 5W^4(R) \text{ on } (0, \infty)$$

We perform a spectral analysis of the operator, which exhibits a unique and simple negative eigenvalue, as well as continuum spectrum; in addition, there is a zero energy resonance. The latter renders the spectral measure singular at zero energy.

- (iv) A contraction argument is set up for ε with a vanishing condition at $\tau = \infty$. For the contraction it is important that N in the first step is sufficiently large.

The first three steps in this paper are essentially the same as in [8]. It is in the final step that we improve on the procedure in [8]. In fact, in Proposition 2.8 of that paper the condition $\nu > \frac{1}{2}$ arises in order to make the embedding

$$(H^{1+2\alpha}(\mathbb{R}^3))^5 \subset H^{2\alpha}(\mathbb{R}^3)$$

for $\nu/2 > 2\alpha \geq \frac{1}{4}$. This was used to control the quintic terms in the construction of the *exact solution* via iteration and application of a suitable parametrix. In fact, the difference shall consist in a more detailed analysis of the *first iterate* for the exact solution, which we exhibit as a sum of two terms,

one of which is smooth, the other of which satisfies a good L^∞ -bound *near the origin*. This latter feature comes from the fact that the loss of smoothness of the approximate solution occurs precisely on the characteristic light cone, and thus one expects the exact solution to be smoother near the spatial origin.

2. THE APPROXIMATE SOLUTION FOR A POWER-LAW RESCALING

2.1. Generalities. In radial coordinates, (1.1) becomes

$$\mathcal{L}_{quintic}u := u_{tt} - u_{rr} - \frac{2}{r}u_r - u^5 = 0 \quad (2.1)$$

This equation is known to be locally well-posed in the space $\mathcal{H} := \dot{H}^1 \times L^2(\mathbb{R}^3)$, meaning that if $(u(0), u_t(0)) \in \mathcal{H}$, then there exists a solution locally in time and continuous in time taking values in \mathcal{H} . Solutions need to be interpreted in the Duhamel sense:

$$u(t) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}u^5(s)ds \quad (2.2)$$

These solutions $\mathcal{L}_{quintic}(u) = 0$ have finite energy:

$$E(u, u_t) = \int_{\mathbb{R}^3} \left[\frac{1}{2}(u_t^2 + |\nabla u|^2) - \frac{u^6}{6} \right] dx = \text{const}$$

A special stationary solution is $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$. By scaling, $\lambda^{\frac{1}{2}}W(\lambda r)$ is also a solution for any $\lambda > 0$. We are interested in letting λ depend on time. More precisely, we would like to find solutions $\mathcal{L}_{quintic}u = 0$ of the form

$$u(t, r) = W_{\lambda(t)}(r) + w(t, r), \quad \lambda(t) \rightarrow \infty \text{ as } t \rightarrow 0+ \quad (2.3)$$

and w small in a suitable sense. It suffices to show that w remains small in energy, since this ensures that the solution blows up at time $t = 0$ by the mechanism of “energy concentration” at the tip of the light-cone $(t, r) = (0, 0)$ (think of solving backwards in time).

2.2. The bulk term. For the convenience of the reader, and in order to correct some minor inaccuracies in [8] such as the omission of harmless $\log R$ factors, we redo the iterative construction from that paper which constitutes step (i) from the four step procedure outlined above. We will henceforth fix $\lambda(t) = t^{-1-\nu}$ with $\nu > 0$. Set

$$u_0(t, r) = \lambda(t)^{\frac{1}{2}}W(r\lambda(t)) = \lambda(t)^{\frac{1}{2}}W(R) \quad (2.4)$$

While u_0 is very far from being an approximate solution, the authors together with D. Tataru showed in [8] that one can add successive corrections

$$u_k = u_0 + v_1 + v_2 + v_3 + \dots + v_k \quad (2.5)$$

so that this function approximately solves (2.1). To be specific, they achieved that $\mathcal{L}_{quintic}u(t)$ goes to zero like t^N in the energy norm restricted to a light

cone where N can be made arbitrarily large by taking k large. This is an iterative construction. Moreover, from the point of view of the energy, the functions v_j are truly lower order, i.e., they will satisfy

$$\int_{r \leq t} |\nabla_{t,x} v_j(t, x)|^2 dx = O(t^\nu) \quad t \rightarrow 0 \quad (2.6)$$

for all $j \geq 1$. In contrast, one of course has

$$\int_{r \leq t} |\nabla_{t,x} u_0(t, x)|^2 dx \simeq 1 \quad t \rightarrow 0$$

We shall now focus on the first two steps of the construction, i.e., $u = u_0 + v_1 + v_2$. Let us compute the error resulting from u_0 . Define $\mathcal{D} := \frac{1}{2} + r\partial_r = \frac{1}{2} + R\partial_R$. Then

$$\begin{aligned} e_0 &:= \mathcal{L}_{quintic} u_0 = \lambda^{\frac{1}{2}}(t) \left[\left(\frac{\lambda'}{\lambda} \right)^2(t) (\mathcal{D}^2 W)(R) + \left(\frac{\lambda'}{\lambda} \right)'(t) (\mathcal{D} W)(R) \right] \\ t^2 e_0 &=: \lambda^{\frac{1}{2}}(t) \left[\omega_1 \frac{1 - R^2/3}{(1 + R^2/3)^{\frac{3}{2}}} + \omega_2 \frac{9 - 30R^2 + R^4}{(1 + R^2/3)^{\frac{5}{2}}} \right] \end{aligned} \quad (2.7)$$

Here ω_j are nonzero constants depending on ν whose values do not concern us.

2.3. The first correction. Then $t^2 e_0 = \lambda(t)^{\frac{1}{2}} O(R^2 \langle R \rangle^{-3})$ as $R \rightarrow \infty$. This error blows up as $t \rightarrow 0$ like t^{-2} . The goal is now to reduce it — in fact turn it into an error that vanishes as $t \rightarrow 0$ — by adding corrections to u_0 , the first one being v_1 . We will do this by setting $\lambda^2(t) L_0 v_1 = e_0$ where

$$L_0 := \partial_R^2 + \frac{2}{R} \partial_R + 5W^4(R) \quad (2.8)$$

Note that this is the linearized operator obtained by plugging $u_0 + v_1$ into (2.1) and discarding ∂_t altogether. While this may seem strange, the idea is to look first at the regime $0 < r \ll t$ where ∂_t should matter less than ∂_r . We shall see shortly that v_1 has the good property that it decays like $(t\lambda(t))^{-2}$, but it produces errors for the nonlinear PDE that grow in r too strongly. To remove this growth, we carry out a correction at the second stage with a differential operator near the light cone $r = t$. There the self-similar variable $a = \frac{r}{t}$ becomes important.

Now we discuss v_1 in more detail. A fundamental system of L_0 is

$$\varphi_1(R) := \frac{1 - R^2/3}{(1 + R^2/3)^{\frac{3}{2}}}, \quad \varphi_2(R) := \frac{1 - 2R^2 + R^4/9}{R(1 + R^2/3)^{\frac{3}{2}}} \quad (2.9)$$

The operator

$$\tilde{L}_0 = R L_0 R^{-1} = \partial_R^2 + 5W^4(R) \quad (2.10)$$

has a fundamental system

$$\begin{aligned}\tilde{\varphi}_1(R) &:= R\varphi_1(R) = \frac{R(1 - R^2/3)}{(1 + R^2/3)^{\frac{3}{2}}} = \tilde{\psi}_1(R^{-2}) \\ \tilde{\varphi}_2(R) &:= R\varphi_2(R) = \frac{1 - 2R^2 + R^4/9}{(1 + R^2/3)^{\frac{3}{2}}} = R\tilde{\psi}_2(R^{-2})\end{aligned}\tag{2.11}$$

The right-hand sides here are for large R , and the $\tilde{\psi}_j$ are analytic around 0. The Wronskian is

$$\tilde{\varphi}_1'(R)\tilde{\varphi}_2(R) - \tilde{\varphi}_1(R)\tilde{\varphi}_2'(R) = 1\tag{2.12}$$

Define $\mu(t) := t\lambda(t)$, and

$$\mu^2(t)L_0v_1 = t^2e_0, \quad v_1(0) = v_1'(0) = 0\tag{2.13}$$

We claim that

$$v_1(t, r) = \mu^{-2}(t)L_0^{-1}t^2e_0 = \lambda^{\frac{1}{2}}(t)\mu^{-2}(t)O(R) \text{ as } R \rightarrow \infty\tag{2.14}$$

To be more specific, write

$$t^2e_0 = \lambda^{\frac{1}{2}}(t)(\omega_1g_1(R) + \omega_2g_2(R))\tag{2.15}$$

see (2.7). Note that the g_j are of the form

$$g_j(R) = R^{-1}\phi_j(R^{-2}) \quad R \gg 1\tag{2.16}$$

where ϕ_j is analytic around 0. Then $L_0f_j = g_j$ with $f_j(0) = f_j'(0) = 0$ satisfies

$$f_j(R) = R^{-1}\left(\tilde{\varphi}_1(R)\int_0^R\tilde{\varphi}_2(R')R'g_j(R')dR' - \tilde{\varphi}_2(R)\int_0^R\tilde{\varphi}_1(R')R'g_j(R')dR'\right)\tag{2.17}$$

for $j = 1, 2$. The one checks that

$$\begin{aligned}f_j(R) &= b_{1j}R + b_{2j} + b_{3j}\frac{\log R}{R} + O(1/R) \text{ as } R \rightarrow \infty \\ f_j(R) &= c_{1j}R^2 + O(R^4) \text{ as } R \rightarrow 0\end{aligned}\tag{2.18}$$

In fact, around $R = 0$ the $f_j(R)$ are even analytic functions, whereas around $R = \infty$ one has the representation

$$\begin{aligned}f_j(R) &= R(b_{1j} + b_{2j}R^{-1} + R^{-2}\log R \varphi_{1j}(R^{-2}) + R^{-2}\varphi_{2j}(R^{-1})) \\ &=: R(F_j(\rho) + \rho^2G_j(\rho^2)\log \rho)\end{aligned}\tag{2.19}$$

where $\varphi_{1j}, \varphi_{2j}$ and F_j, G_j are analytic around zero, with $\rho := R^{-1}$. This follows from (2.11), (2.16), and (2.17). For future reference, we remark that the structure in (2.19) is preserved under application of \mathcal{D} . In particular,

$$v_1(t, r) = \lambda^{\frac{1}{2}}(t)\mu^{-2}(t)(\omega_1f_1(R) + \omega_2f_2(R)) =: \lambda^{\frac{1}{2}}(t)\mu^{-2}(t)f(R)\tag{2.20}$$

Define

$$u_1 := u_0 + v_1 = \lambda^{\frac{1}{2}}(t)(W(R) + \mu^{-2}(t)f(R))$$

In view of (2.18), and $R \leq \mu$ (recall that we are inside of the light cone $r \leq t$)

$$u_1(t, r) = \lambda^{\frac{1}{2}}(t)O(R^{-1}) \quad R \geq 1 \quad (2.21)$$

uniformly in $0 < t < 1$; moreover, we may apply $t\partial_t$ any number of times without affecting this asymptotic property. Finally, $\lambda(t)^{-\frac{1}{2}}u_1(t, r)$ is an even analytic function around $R = 0$.

2.4. The error from u_1 . Set $e_1 := \mathcal{L}_{quintic}(u_1)$. Then

$$e_1 = \partial_t^2 v_1 - 10u_0^3 v_1^2 - 10u_0^2 v_1^3 - 5u_0 v_1^4 - v_1^5 \quad (2.22)$$

One has

$$\begin{aligned} t^2 \lambda^{-\frac{1}{2}}(t)e_1 &= \lambda^{-\frac{1}{2}}(t)((t\partial_t)^2 - t\partial_t)(\lambda^{\frac{1}{2}}(t)w_1(t, r\lambda(t))) - \mu^2(t)(10W^3(R)w_1^2(t, R) \\ &\quad + 10W^2(R)w_1^3(t, R) + 5W(R)w_1^4(t, R) + w_1^5(t, R)) \end{aligned} \quad (2.23)$$

where $w_1(t, R) = \mu^{-2}(t)f(R)$. Then the nonlinearity in (2.23) is

$$\begin{aligned} &\mu^2(t)(10W^3(R)w_1^2(t, R) + 10W^2(R)w_1^3(t, R) + 5W(R)w_1^4(t, R) + w_1^5(t, R)) \\ &= \mu^{-2}(t)(10W^3(R)f^2(R) + 10W^2(R)\mu^{-2}(t)f^3(R) \\ &\quad + 5W(R)\mu^{-4}(t)f^4(R) + \mu^{-6}(t)f^5(R)) \end{aligned} \quad (2.24)$$

whereas

$$\begin{aligned} &\lambda^{-\frac{1}{2}}(t)((t\partial_t)^2 - t\partial_t)(\lambda^{\frac{1}{2}}(t)w_1(t, r\lambda(t))) \\ &= \left(\left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D} \right)^2 - \left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D} \right) \right) w_1(t, R) \\ &= \left((t\partial_t - (1 + \nu)\mathcal{D})^2 - (t\partial_t - (1 + \nu)\mathcal{D}) \right) w_1(t, R) \end{aligned} \quad (2.25)$$

Now $\mu(t) = t^{-\nu}$ whence

$$\mu^2(t) \left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D} \right) \mu^{-2}(t)f(R) = (2\nu - (1 + \nu)\mathcal{D})f(R) \quad (2.26)$$

Note that this is again of the form $f(R)$ with f as in (2.18), (2.19). Thus we can write

$$\begin{aligned} t^2 \lambda^{-\frac{1}{2}}(t)e_1(t, r) &= \mu^{-2}(t) \left(f(R) - (10W^3(R)f^2(R) + 10\mu^{-2}(t)W^2(R)f^3(R) \right. \\ &\quad \left. + 5\mu^{-4}(t)W(R)f^4(R) + \mu^{-6}(t)f^5(R)) \right) \end{aligned} \quad (2.27)$$

We let $a = \frac{r}{t} = \frac{R}{\mu} = Rb$, $b := \mu^{-1}$ and isolate those terms in (2.27) which do not decay for large R . Since we are working inside of the light-cone, we have $0 \leq a \leq 1$. Now, abusing notation somewhat,

$$\begin{aligned}\mu^{-2}(t)f(R) &= b^2R(F(\rho) + \rho^2G(\rho^2)\log\rho) = ba(F(\rho) + \rho^2G(\rho^2)\log\rho) \\ \mu^{-2}(t)W^3(R)f^2(R) &= b^2R^{-3}\Omega(\rho^2)R^2(F(\rho) + \rho^2G(\rho^2)\log\rho)^2 \\ &= b^2R^{-1}(F(\rho) + \rho^2F(\rho)\log\rho + \rho^4G(\rho^2)\log^2\rho) \quad (2.28) \\ \mu^{-4}(t)W^2(R)f^3(R) &= b^4R^{-2}\Omega(\rho^2)R^3(F(\rho) + \rho^2G(\rho^2)\log\rho)^3 \\ &= b^3a(F(\rho) + \rho^2F(\rho)\log\rho + \rho^4F(\rho)\log^2\rho + \rho^6G(\rho^2)\log^3\rho)\end{aligned}$$

where F, G can change from line to line. Similarly,

$$\begin{aligned}\mu^{-6}(t)W(R)f^4(R) &= b^6R^{-1}\Omega(\rho^2)R^4(F(\rho) + \rho^2G(\rho^2)\log\rho)^4 \\ &= b^3a^3(F(\rho) + \rho^2F(\rho)\log\rho + \rho^4F(\rho)\log^2\rho \\ &\quad + \rho^6F(\rho)\log^3\rho + \rho^8G(\rho^2)\log^4\rho) \\ \mu^{-8}(t)f^5(R) &= b^8R^5(F(\rho) + \rho^2G(\rho^2)\log\rho)^5 \\ &= b^3a^5(F(\rho) + \rho^2F(\rho)\log\rho + \rho^4F(\rho)\log^2\rho \\ &\quad + \rho^6F(\rho)\log^3\rho + \rho^8F(\rho)\log^4\rho + \rho^{10}G(\rho^2)\log^5\rho) \quad (2.29)\end{aligned}$$

From (2.28), (2.29) we extract the leading order

$$t^2\lambda^{-\frac{1}{2}}(t)e_1^0(t, r) := \mu^{-1}(t)(c_1a + c_2b + (c_3a + c_4a^3 + c_5a^5)b^2) \quad (2.30)$$

Indeed, from the first line in (2.28) we extract $ba(F(0) + \rho F'(0)) = bac_1 + b^2c_2$, whereas from the fifth we extract $b^3aF(0)$. From the second line in (2.29) we retain $b^3a^3F(0)$, and from the fifth one $b^3a^5F(0)$. The point here is that with this choice of e_1^0 one obtains a decaying error as $R \rightarrow \infty$

$$\begin{aligned}t^2\lambda^{-\frac{1}{2}}(t)(e_1 - e_1^0)(t, r) \\ = \mu^{-2}(t)\left[\frac{\log R}{R}\Phi_1(a, b, \rho\log\rho, \rho) + \frac{1}{R}\Phi_2(a, b, \rho\log\rho, \rho)\right] \quad (2.31)\end{aligned}$$

where $\Phi_j(a, b, u, v)$ are polynomials in a, b and analytic in u, v near $(0, 0)$. Writing $b = \frac{a}{R}$ we may delete the terms involving $b^2 = ba/R$ on the right-hand side of (2.30), since they are of the form (2.31). Thus, it suffices to consider the simpler leading error

$$t^2\lambda^{-\frac{1}{2}}(t)e_1^0(t, r) := c_1a\mu^{-1}(t) + c_2\mu^{-2}(t) = c_1ab + c_2b^2 \quad (2.32)$$

2.5. The second correction. Now we would like to solve the corrector problem “near $r = t''$, i.e.,

$$t^2(v_{tt} - v_{rr} - \frac{2}{r}v_r) = -t^2e_1^0 \quad (2.33)$$

Note that we have discarded the nonlinearity on the left-hand side since it decays near $r = t$. This is designed exactly so as to remove the growth in R . We seek a solution in the form

$$v(t, r) = \lambda(t)^{\frac{1}{2}} (\mu^{-1}(t) q_1(a) + \mu^{-2}(t) q_2(a)) \quad (2.34)$$

with boundary conditions $q_1(0) = 0, q_1'(0) = 0$ and $q_2(0) = 0, q_2'(0) = 0$. These translate into the boundary conditions $v(t, 0) = 0, \partial_r v(t, 0) = 0$. This will essentially be the function v_2 . In view of

$$\lambda(t)^{-\frac{1}{2}} \mu^\alpha \partial_t \lambda(t)^{\frac{1}{2}} \mu^{-\alpha} = \partial_t + \frac{(\alpha - \frac{1}{2})v - \frac{1}{2}}{t}$$

we are reduced to the system

$$t^2 \left(- \left(\partial_t + \frac{v-1}{2t} \right)^2 + \partial_{rr} + \frac{2}{r} \partial_r \right) q_1(a) = c_1 a \quad (2.35)$$

and

$$t^2 \left(- \left(\partial_t + \frac{3v-1}{2t} \right)^2 + \partial_{rr} + \frac{2}{r} \partial_r \right) q_2(a) = c_2 \quad (2.36)$$

Now

$$\begin{aligned} & t^2 \left(- \left(\partial_t + \frac{\beta}{t} \right)^2 + \partial_{rr} + \frac{2}{r} \partial_r \right) f(a) \\ &= ((1 - a^2) \partial_a^2 + (2(\beta - 1)a + 2a^{-1}) \partial_a - \beta^2 + \beta) f(a) \end{aligned}$$

Define

$$L_\beta := (1 - a^2) \partial_a^2 + (2(\beta - 1)a + 2a^{-1}) \partial_a - \beta^2 + \beta \quad (2.37)$$

The system (2.35), (2.36) therefore becomes

$$L_{\frac{v-1}{2}} q_1 = c_1 a, \quad L_{\frac{3v-1}{2}} q_2 = c_2 \quad (2.38)$$

with boundary conditions $q_1(0) = 0, q_1'(0) = 0$ and $q_2(0) = 0, q_2'(0) = 0$.

Since for any integer $n \geq 2$

$$L_\beta a^n = n(n+1)a^{n-2} + (2n(\beta-1) - n(n-1) + \beta - \beta^2)a^n$$

we see that (2.38) has power series solutions

$$q_1(a) = \sum_{j=1}^{\infty} c_{1j} a^{2j+1}, \quad q_2(a) = \sum_{j=1}^{\infty} c_{2j} a^{2j} \quad (2.39)$$

convergent in $|a| < 1$ and unique. In fact, $c_{21} = \frac{c_2}{6}$ and $c_{11} = \frac{c_1}{12}$.

Next, we determine the behavior of these functions as $a \rightarrow 1-$. We remark that in our case, always $\beta > -\frac{1}{2}$. There exists a fundamental system

of L_β of the form

$$\begin{aligned}\psi_1(a) &= 1 + \sum_{\ell=1}^{\infty} d_\ell (1-a)^\ell, \\ \psi_2(a) &= (1-a)^{\beta+1} \left[1 + \sum_{\ell=1}^{\infty} \tilde{d}_\ell (1-a)^\ell \right] = (1-a)^{\beta+1} \psi_2^0(a)\end{aligned}\tag{2.40}$$

provided $\beta \notin \mathbb{Z}_0^+$, see [8, Lemma 3.6]. In that case these series define entire functions. On the other hand, if $\beta \in \mathbb{Z}_0^+$, then ψ_1 is modified to

$$\psi_1(a) = 1 + \sum_{\ell=1}^{\infty} d_\ell (1-a)^\ell + c \psi_2(a) \log(1-a)$$

with a unique choice of c . Notice that in this case the singularity of ψ_1 near $a = 1$ is no worse than $(1-a)^{\beta+1} \log(1-a)$.

In either case the Wronskian

$$W = \psi_1' \psi_2 - \psi_1 \psi_2'$$

satisfies

$$W'(a) = -\frac{2(\beta-1)a + 2a^{-1}}{1-a^2} W(a)$$

whence

$$W(a) = k(1-a^2)^\beta a^{-2}, \quad k \neq 0$$

We define the Green function

$$G_\beta(a, a') := \psi_1(a) \psi_2(a') - \psi_1(a') \psi_2(a)$$

Then a particular solution of $L_\beta q = f$ is given by

$$\begin{aligned}q(a) &= \int_0^a G_\beta(a, \tilde{a}) W(\tilde{a})^{-1} (1-\tilde{a}^2)^{-1} f(\tilde{a}) d\tilde{a} \\ &= k^{-1} \int_0^a G_\beta(a, \tilde{a}) (1-\tilde{a}^2)^{-(\beta+1)} \tilde{a}^2 f(\tilde{a}) d\tilde{a} \\ &= k^{-1} \psi_1(a) \int_0^a \psi_2^0(\tilde{a}) (1+\tilde{a})^{-\beta-1} \tilde{a}^2 f(\tilde{a}) d\tilde{a} \\ &\quad - k^{-1} \psi_2(a) \int_0^a \psi_1(\tilde{a}) (1-\tilde{a}^2)^{-(\beta+1)} \tilde{a}^2 f(\tilde{a}) d\tilde{a}\end{aligned}\tag{2.41}$$

Returning to (2.38) we need to set $f(a) = 1$ and $f(a) = a$ here. Note that $q(a)$ as given by (2.41) satisfies the boundary conditions $q(0) = q'(0) = 0$, whence it agrees with q_2 and q_1 , respectively, as given by (2.39).

Let us first assume that $\beta \notin \mathbb{Z}_0^+$. Then the term on the third line of (2.41) is analytic near $a = 1$. To analyze the behavior of the expression on the fourth line as $a \rightarrow 1$, we note that up to an analytic factor near $a = 1$ it equals

$$(1-a)^{\beta+1} \int_0^a (1-\tilde{a})^{-\beta-1} h(\tilde{a}) d\tilde{a}\tag{2.42}$$

where h is analytic in a neighborhood of $[0, 1]$. By inspection, (2.42) is of the form

$$(1-a)^{\beta+1}((1-a)^{-\beta}H_1(a) + c)H_2(a)$$

where c is a constant and H_1, H_2 are analytic near $[0, 1]$. It follows that $\beta \notin \mathbb{Z}_0^+$, which means that ν is neither an odd positive integer nor of the form $\frac{2n+1}{3}$ with $n \in \mathbb{Z}_0^+$, one has that $q_1(a), q_2(a)$ are analytic in the disk $|a| < 1$ and near $a = 1$ they are of the form

$$Q_1(a) + (1-a)^{\beta+1}Q_2(a) \quad (2.43)$$

where Q_j are analytic near $a = 1$. These functions have the property that after applying ∂_a they remain in $L^2(0, 1)$ due to $\beta > -\frac{1}{2}$. Evidently, they also become smoother as β increases, but they are never infinitely smooth (since β is not an integer).

On the other hand, if $\beta \in \mathbb{Z}_0^+$, then the representation (2.43) needs to be modified with logarithmic factors $\log(1-a)$. We leave these details to the reader.

Using $a = R\mu^{-1}$ we may rewrite (2.34) in the form

$$v_2^0(r, t) := \frac{\lambda(t)^{\frac{1}{2}}}{\mu^2(t)} (R\tilde{q}_1(a) + q_2(a)) \quad (2.44)$$

where we have set $\tilde{q}_1(a) := a^{-1}q_1(a)$. This ensures that both \tilde{q}_1 and q_2 have even power series in a around $a = 0$. Also note that these functions are $O(a^2)$ as $a \rightarrow 0$. We make one more adjustment: in (2.44) one has an odd expansion around $R = 0$, namely just the linear term R . We prefer to modify (2.44) as follows so as to retain the even expansion at $R = 0$:

$$v_2(r, t) := \frac{\lambda(t)^{\frac{1}{2}}}{\mu^2(t)} (R^2 \langle R \rangle^{-1} \tilde{q}_1(a) + q_2(a)) \quad (2.45)$$

Note that for large R this captures the R growth of (2.44), and the next order correction is R^{-1} . With this definition of v_2 we set $u_2 := u_1 + v_2 = u_0 + v_1 + v_2$. By construction, $v_2(t, r)$ is analytic in R around $R = 0$ with an even expansion. Finally, (2.21) remains valid for u_2 as well. In other words, u_0 gives the main shape of the profile as a function of R .

2.6. The error from u_2 . We define

$$\begin{aligned} e_2 &:= \mathcal{L}_{\text{quintic}}(u_2) = \mathcal{L}_{\text{quintic}}(u_1 + v_2) \\ &= \mathcal{L}_{\text{quintic}}(u_1) + u_1^5 - (u_1 + v_2)^5 + (\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r)v_2 \\ &= e_1 - e_1^0 - 5u_1^4v_2 - 10u_1^3v_2^2 - 10u_1^2v_2^3 - 5u_1v_2^4 - v_2^5 \\ &\quad + (\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r)(v_2 - v_2^0) \end{aligned} \quad (2.46)$$

We determine $t^2\lambda(t)^{-\frac{1}{2}}e_2$. First, from (2.31)

$$\begin{aligned} & t^2\lambda^{-\frac{1}{2}}(t)(e_1 - e_1^0)(t, r) \\ &= \mu^{-2}(t) \left[\frac{\log R}{R} \Phi_1(a, b, \rho \log \rho, \rho) + \frac{1}{R} \Phi_2(a, b, \rho \log \rho, \rho) \right] \end{aligned} \quad (2.47)$$

for $R \geq 1$. For $|R| < 1$ we read off from (2.27) and (2.32) that

$$t^2\lambda^{-\frac{1}{2}}(t)(e_1 - e_1^0)(t, r) = O(\mu^{-2}(t)) \quad (2.48)$$

This holds uniformly for small times, and $t\partial_t$ can be applied any number of times without changing this asymptotic behavior as $R \rightarrow 0$.

Next, for large R

$$t^2\lambda^{-\frac{1}{2}}(t)u_1^4v_2 = O(R^{-3}a^2) = O(R^{-1}\mu(t)^{-2})$$

which is of the form (2.47). The final nonlinear term contributes

$$t^2\lambda^{-\frac{1}{2}}(t)v_2^5 = \mu^{-8}O(R^5) = \mu^{-2}R^{-1}O(\mu^{-6}R^6) = O(R^{-1}\mu(t)^{-2})$$

We leave it to the reader to verify that the other nonlinear terms behave in the same fashion. For small R , the nonlinear terms are $O(\mu^{-2}(t)R^2)$.

The difference $v_2 - v_2^0$ contributes this: for $R > 1$,

$$\lambda(t)^{-\frac{1}{2}}t^2(\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r)\frac{\lambda(t)^{\frac{1}{2}}}{\mu(t)^2}((R - R^2\langle R \rangle^{-1})\tilde{q}_1(a)) = O(R^{-1}\mu(t)^{-2}) \quad (2.49)$$

Here we used that $\tilde{q}_1(a) = O(a^2)$ for small a . For small R this term is $O(\mu^{-2}(t)R)$. By inspection, $e_2 = \mathcal{L}_{quintic}(u_2)$ has an even analytic expansion around $R = 0$, and by the preceding we gain a factor μ^{-2} for all R , and the decay is at least $\frac{\log R}{R}$ as $R \rightarrow \infty$.

2.7. Iterating the construction: the next corrections v_3 and v_4 . We now return to Section 2.3 in which we constructed v_1 from e_0 . Here we need to determine v_3 from e_2 via the same route, i.e., by solving $\lambda(t)^2L_0v_3 = e_2$ with zero Cauchy data at $R = 0$. The only essential difference between e_0 and e_2 is that the latter loses a $\log R$ in terms of the R^{-1} decay. The dependence on a, b makes no difference in terms of the asymptotic behavior or the construction, however. Therefore, the same construction as before yields

$$v_3(t, r) = \frac{\lambda(t)^{\frac{1}{2}}}{\mu(t)^4}O(R \log R)$$

as $R \rightarrow \infty$. Moreover, the asymptotic expansion involves terms with both $\log R$ and $\log^2 R$. Around $R = 0$ we have an even Taylor expansion starting off with R^2 . The

$$u_3 := u_0 + v_1 + v_2 + v_3$$

still obeys the decay law (2.21); in fact, any finite number of powers of $\log R$ can be lost since they are more than made up for by the gain of an extra

power of μ^{-2} in v_3 . In this sense, v_3 is of strictly smaller order than both v_1 and v_2 which can be comparable to u_0 on the light-cone $r = t$. For v_4 , one repeats the same construction that lead to v_2 above designed to obtain a decaying error. The error will be on the order of μ^{-4} .

The process can then be repeated any number of times. In [8] the authors, together with D. Tataru, formalized this using various function algebras designed to hold the v_j the associated errors e_j . The details are as follows. We have

$$u_k = v_k + u_{k-1}$$

The error at step k is

$$e_k = (-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u_k + u_k^5$$

we define v_k by

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4\right)v_{2k+1} + e_{2k}^0 = 0 \quad (2.50)$$

respectively

$$\left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right)v_{2k} + e_{2k-1}^0 = 0 \quad (2.51)$$

both equations having zero Cauchy data at $r = 0$. Here at each stage the error term e_k is split into a principal part and a higher order term (to be made precise below),

$$e_k = e_k^0 + e_k^1$$

The successive errors are then computed as

$$e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k}), \quad e_{2k+1} = e_{2k}^1 - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1})$$

where

$$N_{2k+1}(v) = 5(u_{2k}^4 - u_0^4)v + 10u_{2k}^3v^2 + 10u_{2k}^2v^3 + 5u_{2k}v^4 + v^5 \quad (2.52)$$

respectively

$$N_{2k}(v) = 5u_{2k-1}^4v + 10u_{2k-1}^3v^2 + 10u_{2k-1}^2v^3 + 5u_{2k-1}v^4 + v^5 \quad (2.53)$$

The function spaces are as follows. First, the one relevant to the L_0 iteration.

Definition 2.1. $S^m(R^k(\log R)^\ell)$ is the class of smooth functions $v : [0, \infty) \rightarrow \mathbb{R}$ with the following properties:

- (i) v vanishes of order m and $R^{-m}v$ has an even Taylor expansion at $R = 0$.
- (ii) v has a convergent expansion near $R = \infty$ of the form

$$v(R) = \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij} R^{k-i} (\log R)^j$$

The importance of even expansions in R near zero lies with the fact that only those correspond to smooth functions in \mathbb{R}^3 . For the same reason, we will work with even m . Second, we introduce the space arising from the Sturm-Liouville problem near $a = 1$.

Definition 2.2. We define \mathcal{Q} to be the algebra of continuous functions $q : [0, 1] \rightarrow \mathbb{R}$ with the following properties:

- (i) q is analytic in $[0, 1)$ with an even expansion at 0 and with $q(0) = 0$.
- (ii) Near $a = 1$ we have an expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)+1} \sum_{j=0}^{\infty} q_{ij}(a) (\log(1-a))^j$$

with analytic coefficients q_0, q_{ij} ; if ν is irrational, then $q_{ij} = 0$ if $j > 0$. The $\beta(i)$ are of the form

$$\sum_{k \in K} ((2k - 3/2)\nu - 1/2) + \sum_{k \in K'} ((2k - 1/2)\nu - 1/2) \quad (2.54)$$

where K, K' are finite sets of positive integers. Moreover, only finitely many of the q_{ij} are nonzero.

We remark that the exponents of $1-a$ in the above series all exceed $\frac{1}{2}$ because of $\nu > 0$. For the errors e_k we introduce

Definition 2.3. \mathcal{Q}' is the space of continuous functions $q : [0, 1) \rightarrow \mathbb{R}$ with the following properties:

- (i) q is analytic in $[0, 1)$ with an even expansion at 0.
- (ii) Near $a = 1$ we have an expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)} \sum_{j=0}^{\infty} q_{ij}(a) (\log(1-a))^j$$

with analytic coefficients q_0, q_{ij} , of which only finitely many are nonzero. The $\beta(i)$ are as above.

By construction, $\mathcal{Q} \subset \mathcal{Q}'$. The family \mathcal{Q}' is obtained by applying $a^{-1} \partial_a$ to the algebra \mathcal{Q} . The exact number of $\log(1-a)$ factors can of course be determined, but is irrelevant for our purposes.

Definition 2.4. a) $S^m(R^k(\log R)^\ell, \mathcal{Q})$ is the class of analytic functions $v : [0, \infty) \times [0, 1] \times [0, b_0] \rightarrow \mathbb{R}$ so that

- (i) v is analytic as a function of R, b , and

$$v : [0, \infty) \times [0, b_0] \rightarrow \mathcal{Q}$$

- (ii) v vanishes of order m relative to R , and $R^{-m}v$ has an even Taylor expansion at $R = 0$.

- (iii) v has a convergent expansion at $R = \infty$,

$$v(R, a, b) = \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij}(a, b) R^{k-i} (\log R)^j$$

where the coefficients $c_{ij}(\cdot, b) \in \mathcal{Q}$, and $c_{ij}(a, b)$ are analytic in $b \in [0, b_0]$ for all $0 \leq a \leq 1$.

b) $\text{IS}^m(R^k(\log R)^\ell, \mathcal{Q})$ is the class of analytic functions w on the cone C_0 which can be represented as

$$w(r, t) = v(R, a, b), \quad v \in S^m(R^k(\log R)^\ell, \mathcal{Q})$$

The same holds with \mathcal{Q}' in place of \mathcal{Q} .

As in the first two steps we shall exploit that R, a, b are dependent variables, by switching from one representation to another as needed. We shall prove by induction that the successive corrections v_k and the corresponding error terms e_k can be chosen with the following properties. There exist increasing sequences m_k, p_k, q_k of nonnegative integers with $m_1 = p_1 = 0, q_1 = 1$ so that for each $k \geq 1$,

$$v_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k}} \text{IS}^2(R(\log R)^{m_k}, \mathcal{Q}) \quad (2.55)$$

$$t^2 e_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k}} \text{IS}^0(R(\log R)^{p_k}, \mathcal{Q}') \quad (2.56)$$

$$v_{2k} \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k+2}} \text{IS}^2(R^3(\log R)^{p_k}, \mathcal{Q}) \quad (2.57)$$

$$t^2 e_{2k} \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k}} [\text{IS}^0(R^{-1}(\log R)^{q_k}, \mathcal{Q}) + b^2 \text{IS}^0(R(\log R)^{q_k}, \mathcal{Q}')] \quad (2.58)$$

One can of course easily determine the optimal choice of m_k, p_k, q_k from the algorithm outlined below, but this is of no significance. We now inductively verify these claims.

Step 0: *The analysis at $k = 0$*

We observed in (2.7) that

$$t^2 e_0 \in \lambda^{\frac{1}{2}} \text{IS}^0(R^{-1}) \quad (2.59)$$

as claimed. Now assume we know the above relations hold up to $k-1$ with $k \geq 1$, and we show how to construct v_{2k-1} , respectively v_{2k} , so that they hold for the index k .

Step 1: *Begin with e_{2k-2} satisfying (2.58) or (2.59) and choose v_{2k-1} so that (2.55) holds.*

If $k = 1$, then define $e_0^0 := e_0$. If $k > 1$, we use (2.58) to write

$$e_{2k-2} = e_{2k-2}^0 + e_{2k-2}^1$$

where

$$\begin{aligned} t^2 e_{2k-2}^0 &\in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k-2}} \text{IS}^0(R^{-1}(\log R)^{q_{k-1}}, \mathcal{Q}), \\ t^2 e_{2k-2}^1 &\in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k}} \text{IS}^0(R(\log R)^{q_{k-1}}, \mathcal{Q}') \end{aligned} \quad (2.60)$$

We note that the term e_{2k-2}^1 can be included in e_{2k-1} , cf. (2.56). We define v_{2k-1} as in (2.50) neglecting the a dependence of e_{2k-2}^0 . In other words, we choose to treat a as a parameter and the error resulting from this choice will then be incorporated in e_{2k-1} .

Changing variables to R in (2.50) we need to solve the equation

$$\mu(t)^2 L_0 v_{2k-1} = -t^2 e_{2k-2}^0 \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^{2k-2}} \text{IS}^0(R^{-1}(\log R)^{q_{k-1}}, \mathcal{Q})$$

where the operator L_0 is the one from above. Then (2.55) is a consequence of the following ODE lemma. We remark that the statement of Lemma 2.1 is not optimal with respect to the number of logarithms in R . But for the sake of simplicity we choose this formulation.

Lemma 2.1. *The solution v to the equation*

$$L_0 v = f \in S^0(R^{-1}(\log R)^\ell), \quad v(0) = v'(0) = 0$$

with integer $\ell \geq 0$ has the regularity

$$v \in S^2(R(\log R)^\ell) \quad (2.61)$$

Proof. This follows from the representation (2.11), (2.16), and (2.17). Indeed,

$$\begin{aligned} v(R) &= R^{-1} \left(\tilde{\varphi}_1(R) \int_0^R \tilde{\varphi}_2(R') R' f(R') dR' - \tilde{\varphi}_2(R) \int_0^R \tilde{\varphi}_1(R') R' f(R') dR' \right) \\ &= R^{-1} \tilde{\psi}_1(R^{-2}) \int_{R_0}^R y \tilde{\psi}_2(y^{-2}) \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij} y^{-i} (\log y)^j dy \\ &\quad - \tilde{\psi}_2(R^{-2}) \int_{R_0}^R \tilde{\psi}_1(y^{-2}) \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij} y^{-i} (\log y)^j dy \\ &\quad + c_1 \phi_1(R) + c_2 \phi_2(R) \end{aligned} \quad (2.62)$$

The first integral in (2.62) contributes

$$\begin{aligned} &R^{-1} \tilde{\psi}_1(R^{-2}) \int_{R_0}^R \sum_{n=0}^{\infty} a_n y^{-2n} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij} y^{1-i} (\log y)^j dy \\ &= \text{const } \phi_1(R) + \tilde{\psi}_1(R^{-2}) \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} \chi_{[1-2n-i \neq -1]} a_{nij} R^{1-i-2n} (\log R)^j \\ &\quad + R^{-1} \tilde{\psi}_1(R^{-2}) \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} \chi_{[1-2n-i = -1]} a_{nij} (\log R)^{j+1} \end{aligned}$$

which lies in $S^2(R(\log R)^\ell)$, whereas the second one contributes

$$\begin{aligned} & \tilde{\psi}_2(R^{-2}) \int_{R_0}^R \sum_{n=0}^{\infty} b_n y^{-2n} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} c_{ij} y^{-i} (\log y)^j dy \\ &= \text{const } \phi_2(R) + \tilde{\psi}_2(R^{-2}) \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} \chi_{[-2n-i \neq -1]} b_{nij} R^{1-i-2n} (\log R)^j \\ &+ \tilde{\psi}_2(R^{-2}) \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\ell+i} \chi_{[-2n-i = -1]} b_{nij} (\log R)^{j+1} \end{aligned}$$

which again lies in $S^2(R(\log R)^\ell)$. □

We remark that v_1 as constructed above satisfies

$$v_1 \in \frac{\lambda^{\frac{1}{2}}}{\mu(t)^2} S^2(R),$$

see (2.18).

Step 2: Show that if v_{2k-1} is chosen as above then (2.56) holds.

Thinking of v_{2k-1} as a function of t, R and a we can write e_{2k-1} in the form

$$e_{2k-1} = N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

Here $N_{2k-1}(v_{2k-1})$ accounts for the contribution from the nonlinearity and is given by (2.52). $E^t v_{2k-1}$ contains the terms in

$$-\partial_{tt} v_{2k-1}(t, R, a) \tag{2.63}$$

where no derivative applies to the variable a , while $E^a v_{2k-1}$ contains those terms in

$$\left(\partial_{tt} - \partial_{rr} - \frac{2}{r} \partial_r \right) v_{2k-1}(t, R, a)$$

where at least one derivative applies to the variable a (recall that in Step 1 the parameter a was frozen). We begin with the terms in N_{2k-1} . We first note that, by summing the v_j over $1 \leq j \leq 2k-2$,

$$u_{2k-2} - u_0 \in \frac{\lambda^{\frac{1}{2}}}{\mu^2} \text{IS}^2(R(\log R)^n, Q) \tag{2.64}$$

for some integer $n = n(k) \geq 0$. The first term in $N_{2k-1}(v_{2k-1})$ contributes

$$\begin{aligned} t^2(u_{2k-2}^4 - u_0^4)v_{2k-1} &= t^2[(u_{2k-2} - u_0)^4 + 4(u_{2k-2} - u_0)^3 u_0 \\ &+ 6(u_{2k-2} - u_0)^2 u_0^2 + 4(u_{2k-2} - u_0) u_0^3] v_{2k-1} \end{aligned} \tag{2.65}$$

Using (2.64) we compute

$$\begin{aligned}
& t^2(u_{2k-2} - u_0)^4 v_{2k-1} \\
& \in \frac{1}{(t\lambda)^6} \text{IS}^8(R^4 (\log R)^{4n}, \mathcal{Q}) \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R (\log R)^{m_k}, \mathcal{Q}) \\
& \subset a^6 \text{IS}^2(R^{-2} (\log R)^{4n}, \mathcal{Q}) \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \text{IS}^2(R (\log R)^{m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \text{IS}^2(R^{-1} (\log R)^{p_k}, \mathcal{Q})
\end{aligned}$$

as well as

$$\begin{aligned}
& t^2(u_{2k-2} - u_0) u_0^3 v_{2k-1} \\
& \in t^2 \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2} \text{IS}^2(R (\log R)^n, \mathcal{Q}) \lambda^{\frac{3}{2}} S^0(R^{-3}) \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R (\log R)^{m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R^{-1} (\log R)^{p_k}, \mathcal{Q})
\end{aligned}$$

The other two terms in (2.65) are similar. Next, compute

$$\begin{aligned}
t^2 v_{2k-1}^5 & \in \frac{t^2 \lambda^{\frac{5}{2}}}{(t\lambda)^{10k}} \text{IS}^{10}(R^5 (\log R)^{5m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}} R^6}{(t\lambda)^{10k-2}} \text{IS}^4(R^{-1} (\log R)^{5m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} a^6 b^{8(k-1)} \text{IS}^2(R^{-1} (\log R)^{5m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R^{-1} (\log R)^{p_k}, \mathcal{Q})
\end{aligned}$$

and

$$\begin{aligned}
& t^2 u_{2k-2}^3 v_{2k-1}^2 \\
& \in \lambda^{-\frac{1}{2}} (t\lambda)^2 \text{IS}^0(R^{-3}, \mathcal{Q}) \frac{\lambda}{(t\lambda)^{4k}} \text{IS}^4(R^2 (\log R)^{2m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} b^{2k-2} \text{IS}^4(R^{-1} (\log R)^{2m_k}, \mathcal{Q}) \\
& \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R^{-1} (\log R)^{p_k}, \mathcal{Q})
\end{aligned}$$

with similar statements for $u_{2k-2}^2 v_{2k-1}^3$ and $u_{2k-2} v_{2k-1}^4$. Summing up we obtain

$$N_{2k-1}(v_{2k-1}) \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R^{-1}(\log R)^{p_k}, \mathcal{Q}) \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R^{-1}(\log R)^{p_k}, \mathcal{Q}')$$

This concludes the analysis of $N_{2k-1}(v_{2k-1})$. We continue with the terms in $E^t v_{2k-1}$, where we can neglect the a dependence. Therefore, it suffices to compute

$$t^2 \partial_t^2 \left(\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R(\log R)^{m_k}) \right) \subset \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R(\log R)^{m_k})$$

Finally, we consider the terms in $E^a v_{2k-1}$. With

$$v_{2k-1}(r, t) = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} w(R, a), \quad w \in S^2(R(\log R)^{m_k}, \mathcal{Q})$$

we have

$$\begin{aligned} t^2 E^a v_{2k-1} &= -2t \partial_t \left(\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \right) a w_a(R, a) + \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} [2(\nu + 1) a R w_{aR}(R, a) \\ &\quad - 2R a^{-1} w_{Ra} - 2a^{-1} w_a(R, a) + (a^2 - 1) w_{aa}(R, a) + 2a w_a(R, a)] \end{aligned}$$

Since \mathcal{Q} are even in a we conclude that

$$(1 - a^2) \partial_{aa}, a \partial_a, a^{-1} \partial_a : \mathcal{Q} \rightarrow \mathcal{Q}'$$

and therefore

$$t^2 E^a v_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \text{IS}^2(R(\log R)^{m_k}, \mathcal{Q}')$$

This concludes the proof of (2.56).

Step 3: Define v_{2k} so that (2.57) holds.

We begin the analysis with e_{2k-1} replaced by its main asymptotic component at $R = \infty$:

$$\begin{aligned} t^2 f_{2k-1} &= \frac{\lambda^{\frac{1}{2}}}{\mu^{2k-1}} \left(\sum_{j=0}^{p_k} a q_j(a) (\log R)^j + b \sum_{j=0}^{p_k+1} [\tilde{q}_j^1(a) + a \tilde{q}_j^2(a)] (\log R)^j \right. \\ &\quad \left. + b^2 \sum_{j=0}^{p_k+1} \tilde{\tilde{q}}_j(a) (\log R)^j \right), \quad q_j, \tilde{q}_j^{1,2}, \tilde{\tilde{q}}_j \in \mathcal{Q}' \end{aligned} \tag{2.66}$$

This is chosen such that $t^2(e_{2k-1} - f_{2k-1})$ consists of terms either decaying at least like $R^{-1}(\log R)^{p_k+1}$, or else gaining at least a factor b^2 . Consider the equation (2.51) with f_{2k-1} on the right-hand side,

$$t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) \tilde{v}_{2k} = -t^2 f_{2k-1}$$

Homogeneity considerations suggest that we should look for a solution \tilde{v}_{2k} which has the form

$$\begin{aligned} \tilde{v}_{2k} = \frac{\lambda^{\frac{1}{2}}}{\mu^{2k-1}} & \left(\sum_{j=0}^{p_k} W_{2k}^j(a) (\log R)^{j+b} \sum_{\kappa=1,2} \sum_{j=0}^{p_k+1} \tilde{W}_{2k}^{j,\kappa}(a) (\log R)^j \right. \\ & \left. + b^2 \sum_{j=0}^{p_k+1} \widetilde{\widetilde{W}}_{2k}^j(a) (\log R)^j \right) \end{aligned}$$

The one-dimensional equations for W_{2k}^j are obtained by matching the powers of $\log R$. This gives the system of equations

$$\begin{aligned} t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) & \left(\frac{\lambda^{\frac{1}{2}}}{\mu^{2k-1}} W_{2k}^j(a) \right) = \frac{\lambda^{\frac{1}{2}}}{\mu^{2k-1}} (aq_j(a) - F_j(a)) \\ t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) & \left(\frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \tilde{W}_{2k}^{j,\kappa}(a) \right) = \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} (a^{\kappa-1} \tilde{q}_j^\kappa(a) - \tilde{F}_j^\kappa(a)), \quad \kappa = 1, 2, \\ t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) & \left(\frac{\lambda^{\frac{1}{2}}}{\mu^{2k+1}} \widetilde{\widetilde{W}}_{2k}^j(a) \right) = \frac{\lambda^{\frac{1}{2}}}{\mu^{2k+1}} (\tilde{\tilde{q}}_j(a) - \tilde{\tilde{F}}_j(a)) \end{aligned}$$

where

$$\begin{aligned} F_j(a) &= (j+1) \left[((\nu+1)\nu(2k-1) + 2a^{-2}) W_{2k}^{j+1} + (a^{-1} - (1+\nu)a) \partial_a W_{2k}^{j+1} \right] \\ &\quad + (j+2)(j+1)((\nu+1)^2 + a^{-2}) W_{2k}^{j+2} \\ \tilde{F}_j^\kappa(a) &= (j+1) \left[(2k(\nu+1)\nu + 2a^{-2}) \tilde{W}_{2k}^{j+1,\kappa} + (a^{-1} - (1+\nu)a) \partial_a \tilde{W}_{2k}^{j+1,\kappa} \right] \\ &\quad + (j+2)(j+1)((\nu+1)^2 + a^{-2}) \tilde{W}_{2k}^{j+2,\kappa} \\ \tilde{\tilde{F}}_j(a) &= (j+1) \left[((\nu+1)\nu(2k+1) + 2a^{-2}) \widetilde{\widetilde{W}}_{2k}^{j+1} + (a^{-1} - (1+\nu)a) \partial_a \widetilde{\widetilde{W}}_{2k}^{j+1} \right] \\ &\quad + (j+2)(j+1)((\nu+1)^2 + a^{-2}) \widetilde{\widetilde{W}}_{2k}^{j+2} \end{aligned} \tag{2.67}$$

Here we make the convention that $W_{2k}^j = 0$ for $j > p_k$ and $\tilde{W}_{2k}^{j,\kappa} = \widetilde{\widetilde{W}}_{2k}^j = 0$ for $j > p_k + 1$. Then we solve the equations in this system successively for decreasing values of j from p_k to 0, respectively $p_k + 1$ to 0.

Conjugating out the power of t we get

$$\begin{aligned} t^2 \left(- \left(\partial_t + \frac{(2k-3/2)\nu-1/2}{t} \right)^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) W_{2k}^j(a) &= a q_j(a) - F_j(a) \\ t^2 \left(- \left(\partial_t + \frac{(2k-1/2)\nu-1/2}{t} \right)^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) \tilde{W}_{2k}^{j,\kappa}(a) &= a^{\kappa-1} \tilde{q}_j^\kappa(a) - \tilde{F}_j^\kappa(a) \\ t^2 \left(- \left(\partial_t + \frac{(2k+1/2)\nu-1/2}{t} \right)^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) \widetilde{\tilde{W}}_{2k}^j(a) &= \tilde{\tilde{q}}_j(a) - \tilde{\tilde{F}}_j(a) \end{aligned}$$

which we rewrite as an equation in the a variable,

$$\begin{aligned} L_\beta W_{2k}^j &= a q_j(a) - F_j(a), \quad \beta = (2k-3/2)\nu-1/2 \\ L_\beta \tilde{W}_{2k}^{j,\kappa} &= a^{\kappa-1} \tilde{q}_j^\kappa(a) - \tilde{F}_j^\kappa(a), \quad \beta = (2k-1/2)\nu-1/2 \\ L_\beta \widetilde{\tilde{W}}_{2k}^j &= \tilde{\tilde{q}}_j(a) - \tilde{\tilde{F}}_j(a), \quad \beta = (2k+1/2)\nu-1/2 \end{aligned} \quad (2.68)$$

where the one-parameter family of operators L_β is defined as above, i.e.,

$$L_\beta = (1-a^2)\partial_a^2 + 2(a^{-1} + a\beta - a)\partial_a - \beta^2 + \beta \quad (2.69)$$

see (2.37). We claim that solving this system with 0 Cauchy data at $a = 0$ yields solutions which satisfy

$$\begin{aligned} W_{2k}^j &\in a \mathcal{Q}, \quad j = \overline{0, p_k} \\ \tilde{W}_{2k}^{j,\kappa} &\in a^{\kappa-1} \mathcal{Q}, \quad j = \overline{0, p_k+1}, \kappa = 1, 2, \\ \widetilde{\tilde{W}}_{2k}^j &\in \mathcal{Q}, \quad j = \overline{0, p_k+1} \end{aligned} \quad (2.70)$$

The choice of β in (2.68) explains the appearance of the set (2.54). The fact that we need integer multiples of the exponents in (2.68) is a result of the nonlinearity which requires taking powers (moreover, \mathcal{Q} is not an algebra otherwise).

The claim (2.70) is established as in the computation of v_2 above, see [8, Lemma 3.6] and [7, Lemma 3.9] for details.

As in the case of v_2 , we need to make some adjustments. First, we modify \tilde{v}_{2k} to ensure an even expansion¹ around $R = 0$:

$$\begin{aligned} \tilde{v}_{2k} &= \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \left(R^2 \langle R \rangle^{-1} \sum_{j=0}^{p_k} a^{-1} W_{2k}^j(a) (\log R)^j + \sum_{j=0}^{p_k+1} \tilde{W}_{2k}^{j,1}(a) (\log R)^j \right. \\ &\quad \left. + R^2 \langle R \rangle^{-1} \sum_{j=0}^{p_k+1} a^{-1} \tilde{W}_{2k}^{j,2}(a) (\log R)^j + b \sum_{j=0}^{p_k+1} \widetilde{\tilde{W}}_{2k}^j(a) (\log R)^j \right) \end{aligned}$$

¹We use the notation $\langle R \rangle = \sqrt{1+R^2}$.

This is not admissible because of the singularity of $\log R$ at $R = 0$. We therefore modify this expression further:

$$\begin{aligned} v_{2k} &= \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \left(R^2 \langle R \rangle^{-1} \sum_{j=0}^{p_k} a^{-1} W_{2k}^j(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j + \sum_{j=0}^{p_k+1} \tilde{W}_{2k}^{j,1}(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right. \\ &\quad \left. + R^2 \langle R \rangle^{-1} \sum_{j=0}^{p_k+1} a^{-1} \tilde{W}_{2k}^{j,2}(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j + b \sum_{j=0}^{p_k+1} \tilde{\tilde{W}}_{2k}^j(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right) \end{aligned}$$

Step 4: For v_{2k} defined as above we show that the corresponding error e_{2k} satisfies (2.58).

We can write e_{2k} in the form

$$t^2 e_{2k} = t^2 \left(e_{2k-1} - e_{2k-1}^0 \right) + t^2 \left(e_{2k-1}^0 - \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r} \partial_r \right) v_{2k} \right) + t^2 N_{2k}(v_{2k})$$

where N_{2k} is defined by (2.53) and

$$\begin{aligned} t^2 e_{2k-1}^0 &= \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \left(\sum_{j=0}^{p_k} R^2 \langle R \rangle^{-1} q_j(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right. \\ &\quad \left. + \sum_{j=0}^{p_k+1} \tilde{q}_j^1(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right. \\ &\quad \left. + b \sum_{j=0}^{p_k+1} R^2 \langle R \rangle^{-1} \tilde{q}_j^2(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right. \\ &\quad \left. + b \sum_{j=0}^{p_k+1} \tilde{\tilde{q}}_j(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right), \end{aligned} \tag{2.71}$$

with $q_j, \tilde{q}_j^{1,2}, \tilde{\tilde{q}}_j(a) \in \mathcal{Q}'$. We begin with the first term in e_{2k} , which has the form

$$t^2 (e_{2k-1} - e_{2k-1}^0) \in \frac{\lambda^{\frac{1}{2}}}{\mu^{2k}} \left[\text{IS}^0(R^{-1}(\log R)^{p_k+2}, \mathcal{Q}') + b^2 \text{IS}^0(R(\log R)^{p_k}, \mathcal{Q}') \right]$$

The second term is admissible for (2.58). It remains to show that

$$\text{IS}^0(R^{-1}(\log R)^{q_k}, \mathcal{Q}') \subset \text{IS}^0(R^{-1}(\log R)^{q_k}, \mathcal{Q}) + b^2 \text{IS}^0(R(\log R)^{q_k}, \mathcal{Q}') \tag{2.72}$$

which can be seen as follows: for $w \in \text{IS}^0(R^{-1}(\log R)^{q_k}, \mathcal{Q}')$ we write

$$w = (1 - a^2)w + \frac{1}{(t\lambda)^2} R^2 w$$

Then

$$(1 - a^2)w \in \text{IS}^0(R^{-1}(\log R)^{q_k}, \mathcal{Q}), \quad \frac{1}{(t\lambda)^2} R^2 w \in b^2 \text{IS}^0(R(\log R)^{q_k}, \mathcal{Q}')$$

as desired. The second term in e_{2k} would equal 0 if we were to replace $\frac{1}{2} \log(1 + R^2)$ by $\log R$ in both e_{2k}^0 and v_{2k} ; in addition, as in the case of e_2 above, error terms arise due to the replacement of R with $R^2 \langle R \rangle^{-1}$. We leave those latter terms to the reader. To illustrate the former, consider the difference which is obtained upon replacing the derivatives of $\frac{1}{2} \log(1 + R^2)$ by derivatives of $\log R$ in the expression

$$t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r} \partial_r \right) \left(\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} \sum_j W_{2k}^j(a) \left(\frac{1}{2} \log(1 + R^2) \right)^j \right)$$

Computing these differences one finds that the second term in e_{2k} is a sum of expressions of the form

$$\begin{aligned} & \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} \sum_j \left(\frac{W_{2k}^j(a)}{a^2} \left[S(R^{-2})(\log(1 + R^2))^{j-1} + S(R^{-2})(\log(1 + R^2))^{j-2} \right] \right. \\ & \quad \left. + \frac{\partial_a W_{2k}^j(a)}{a} S(R^{-2})(\log(1 + R^2))^{j-1} \right) \end{aligned}$$

Since W_{2k}^j are cubic at 0 it follows that we can pull out an a factor and see that this part of the error is in

$$\frac{1}{(t\lambda)^{2k}} \text{IS}^1(R^{-1}(\log R)^{q_k}, \mathcal{Q}')$$

which is admissible by (2.72). The nonlinear expression is left to the reader.

In summary we arrive at the following result.

Proposition 2.2. *For any $N \geq 1$ there exists a positive integer k such that u_k as constructed above satisfies*

$$\int_{r \leq t} |(\mathcal{L}_{\text{quintic}} u_k)(r, t)|^2 r^2 dr = O(t^N) \quad t \rightarrow 0+$$

Moreover, $u_k - W_{\lambda(t)} \in \frac{\lambda^{\frac{1}{2}}}{\mu^2} O(R)$, with $R = r\lambda$.

The following proposition is going to be the key for the proof of Theorem 1.1. It states that the approximate solutions of the previous proposition can be turned into actual solutions inside the light cone.

Proposition 2.3. *Fix $N > 1$ sufficiently large. Then there is $k \geq 1$ sufficiently large and there exists a function $\varepsilon(t, r)$ with*

$$\varepsilon(t, \cdot) \in t^N H_{R^2 dR}^{1+\frac{N}{2}-}, \quad \varepsilon_t(t, \cdot) \in t^N H_{R^2 dR}^{\frac{N}{2}-}$$

such that the function

$$u(t, r) = u_{2k-1}(t, r) + \varepsilon(t, r)$$

is a solution of (1.1) inside the light cone.

Proof of Theorem 1.1 assuming Proposition 2.3. As in [8] one observes that u_k can be extended outside the light cone such that

$$\int_{r>t} [|\nabla_x u_k|^2 + |\nabla_x W_{\lambda(t)}|^2 + (\partial_t W_{\lambda(t)})^2 + |\nabla_t u_k|^2] dx < \delta$$

for any prescribed $\delta > 0$, provided $0 < t < t_0$ with $t_0 = t_0(\delta, k, \nu)$ sufficiently small. But then, picking t_0 sufficiently small, we can arrange that the $\dot{H}^1 \times L^2$ -norm of $(u(t_0, \cdot), u_t(t_0, \cdot))$ is less than δ in the region $r > t_0$. Then Huyghen's principle and the small energy global regularity imply that the corresponding solution remains of class $H^{1+\frac{\nu}{2}-}$ on the exterior of the light cone. \square

The remainder of the paper is devoted to proving Proposition 2.3. The idea is that up to smoother errors, the principal source term e_{2k-1} (we take u_{2k-1} to be the approximate solution) may be reduced to an explicit algebraic expression. In fact, due to Huyghen's principle, we can and shall modify e_{2k-1} outside the light cone to simplify the analysis a bit. We shall then see that the error term $\varepsilon(t, r)$, which will be constructed via a suitable iteration scheme, can be written as a sum of a smoother and a "small" term (in the sense of amplitude). This decomposition in fact results naturally from the structure of the first iterate of the scheme used to construct $\varepsilon(t, r)$.

3. SETTING UP THE ITERATION SCHEME; FORMULATION ON THE FOURIER SIDE

Introduce the variables $\tau = \nu^{-1}t^{-\nu}$, $R = \lambda(t)r$, and write

$$\tilde{\varepsilon}(\tau, R) := R\varepsilon(t(\tau), r(\tau, R))$$

Then we get the following² equation for $\tilde{\varepsilon}$:

$$\begin{aligned} & (\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2 \tilde{\varepsilon} - (\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R) \tilde{\varepsilon} + \mathcal{L}\tilde{\varepsilon} \\ & = \lambda^{-2}(\tau)R[N_{2k-1}(\varepsilon) + e_{2k-1}] + \partial_\tau(\dot{\lambda}\lambda^{-1})\tilde{\varepsilon}; \end{aligned} \tag{3.1}$$

where the operator \mathcal{L} is given by

$$\mathcal{L} = -\partial_R^2 - 5W^4(R)$$

and we have

$$\begin{aligned} RN_{2k-1}(\varepsilon) &= 5(u_{2k-1}^4 - u_0^4)\tilde{\varepsilon} + RN(u_{2k-1}, \tilde{\varepsilon}), \\ RN(u_{2k-1}, \tilde{\varepsilon}) &= R(u_{2k-1} + \frac{\tilde{\varepsilon}}{R})^5 - Ru_{2k-1}^5 - 5u_{2k-1}^4\tilde{\varepsilon} \end{aligned}$$

²In fact, the term $\partial_\tau(\dot{\lambda}\lambda^{-1})\tilde{\varepsilon}$ was inadvertently omitted in [8], which has no bearing on the proof there, however.

Introducing the operator, with $\beta_v(\tau) = \frac{\dot{\lambda}(\tau)}{\lambda(\tau)}$,

$$\mathcal{D} = \partial_\tau + \beta_v(\tau)(R\partial_R - 1),$$

we can also write the above equation as

$$\mathcal{D}^2 \tilde{\varepsilon} + \beta_v(\tau) \mathcal{D} \tilde{\varepsilon} + \mathcal{L} \tilde{\varepsilon} = \lambda^{-2}(\tau) [5(u_{2k-1}^4 - u_0^4) \tilde{\varepsilon} + RN(u_{2k-1}, \tilde{\varepsilon}) + Re_{2k-1}] \quad (3.2)$$

In the following, we shall freely borrow the facts about the spectral theory and associated distorted Fourier transform contained in [8] as well as [2]. In particular, we recall that there exists a Fourier basis $\phi(R, \xi)$ and associated spectral measure $\rho(\xi)$ satisfying the asymptotic expansions and growth conditions explained in [8, Section 4] such that

$$\tilde{\varepsilon}(\tau, R) = x_d(\tau) \phi_d(R) + \int_0^\infty x(\tau, \xi) \phi(R, \xi) \rho(\xi) d\xi$$

For the asymptotic behavior of $\phi(R, \xi)$ in various regimes see (4.2) and (4.4). Here the functions $x(\tau, \xi)$ are the (distorted) Fourier coefficients associated with $\tilde{\varepsilon}$, and $\phi_d(R)$ is the unique ground state with associated negative eigenvalue for the operator \mathcal{L} . We also note the important asymptotic estimates

$$\rho(\xi) \simeq \xi^{-\frac{1}{2}}, \quad \xi \ll 1, \quad \rho(\xi) \simeq \xi^{\frac{1}{2}}, \quad \xi \gg 1. \quad (3.3)$$

as well as the fact that near $\xi = 0$ as well as $\xi = \infty$ the spectral measure behaves like a symbol upon differentiation. We shall henceforth write

$$\underline{x}(\tau, \xi) := \begin{pmatrix} x_d(\tau) \\ x(\tau, \xi) \end{pmatrix}, \quad \underline{\xi} = \begin{pmatrix} \xi_d \\ \xi \end{pmatrix}$$

Then proceeding as in [2], in particular section 3.5 in loc. cit. which uses a variation on the procedure in [8], we derive the following transport equation for $x(\tau, \xi)$:

$$(\mathcal{D}_\tau^2 + \beta_v(\tau) \mathcal{D}_\tau + \underline{\xi}) \underline{x}(\tau, \xi) = \mathcal{R}(\tau, \underline{x}) + f(\tau, \underline{\xi}), \quad (3.4)$$

where we have

$$\mathcal{R}(\tau, \underline{x})(\xi) = \left(-4\beta_v(\tau) \mathcal{K} \mathcal{D}_\tau \underline{x} - \beta_v^2(\tau) (\mathcal{K}^2 + [\mathcal{A}, \mathcal{K}] + \mathcal{K} + \beta'_v \beta_v^{-2} \mathcal{K}) \underline{x} \right)(\xi) \quad (3.5)$$

with $\beta_v(\tau) = \frac{\dot{\lambda}(\tau)}{\lambda(\tau)}$, and

$$f(\tau, \xi) = \mathcal{F}(\lambda^{-2}(\tau) [5(u_{2k-1}^4 - u_0^4) \tilde{\varepsilon} + RN(u_{2k-1}, \tilde{\varepsilon}) + Re_{2k-1}]) (\xi) \quad (3.6)$$

Also the key operator

$$\mathcal{D}_\tau = \partial_\tau + \beta_v(\tau) \mathcal{A}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_c \end{pmatrix}$$

and we have

$$\mathcal{A}_c = -2\xi \partial_\xi - \left(\frac{5}{2} + \frac{\rho'(\xi) \xi}{\rho(\xi)} \right)$$

Finally, we observe that the “transference operator” \mathcal{K} is given by the following type of expression

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{dd} & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_{cc} \end{pmatrix} \quad (3.7)$$

with mapping properties specified later on.

In the following, we shall take advantage of the observation that the equation

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}(\tau, \xi) = \underline{f}(\tau, \xi) = \begin{pmatrix} f_d(\tau) \\ f(\tau, \xi) \end{pmatrix}$$

can be solved completely explicitly; in particular, imposing vanishing boundary data at $\tau = \infty$, we obtain the following expression for the continuous part $x(\tau, \xi)$:

$$x(\tau, \xi) = \xi^{-\frac{1}{2}} \int_\tau^\infty \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \sin \left[\lambda(\tau)\xi^{\frac{1}{2}} \int_\tau^\sigma \lambda^{-1}(u) du \right] f(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi) d\sigma \quad (3.8)$$

We shall justify this below. On the other hand, one immediately obtains the elementary implicit relation

$$x_d(\tau) = \int_\tau^\infty H_d(\tau, \sigma) \tilde{f}_d(\sigma) d\sigma, \quad H_d(\tau, \sigma) = -\frac{1}{2} |\xi_d|^{-\frac{1}{2}} e^{-|\xi_d|^{\frac{1}{2}} |\tau - \sigma|} \quad (3.9)$$

$$\tilde{f}_d(\sigma) = f_d(\sigma) - \beta_\nu(\sigma) \partial_\tau x_d(\sigma)$$

To derive (3.8), define an operator

$$(Mf)(\tau, \xi) := \lambda^{-\frac{5}{2}}(\tau) \rho(\xi)^{\frac{1}{2}} f(\tau, \xi)$$

and $\mathcal{D}_1 := \partial_\tau - 2\beta_\nu(\tau)\xi\partial_\xi$. Then

$$M^{-1}\mathcal{D}_1 M = \mathcal{D} := \partial_\tau + \beta_\nu(\tau)\mathcal{A}_c$$

Second, define

$$(Sf)(\tau, \xi) := f(\tau, \lambda^{-2}(\tau)\xi)$$

Then $S^{-1}\partial_\tau S = \mathcal{D}_1$ whence

$$M^{-1}S^{-1}\partial_\tau SM = \mathcal{D}$$

It follows that

$$(SM)^{-1}(\partial_\tau^2 + \beta_\nu(\tau)\partial_\tau + \lambda^{-2}(\tau)\xi)SM = \mathcal{D}^2 + \beta_\nu(\tau)\mathcal{D} + \xi \quad (3.10)$$

Finally, one checks that $H(\tau, \xi) := \sin(\xi^{\frac{1}{2}}\omega(\tau))$, $\omega(\tau) := \int^\tau \lambda^{-1}$ satisfies

$$(\partial_\tau^2 + \beta_\nu(\tau)\partial_\tau + \lambda^{-2}(\tau)\xi)H(\tau, \xi) = 0$$

Hence,

$$H(\xi, \sigma, \tau) = \xi^{-\frac{1}{2}} \sin \left(\xi^{\frac{1}{2}} \int_\tau^\sigma \lambda^{-1}(u) du \right), \quad \sigma \geq \tau$$

is the fundamental solution of the operator in the parentheses on the left-hand side of (3.10). In order to measure the size of x , we use the norms

$$\|x\|_{L_{d\rho}^{2,\alpha}} = \|\langle \xi \rangle^\alpha x\|_{L_{d\rho}^2}$$

The following lemma shall be used throughout the sequel without further mention:

Lemma 3.1. *Assume N is sufficiently large and write*

$$\|f\|_{L_{d\rho}^{2,\alpha;N}} := \sup_{\tau > \tau_0} \tau^N (\|f(\tau, \cdot)\|_{L_{d\rho}^{2,\alpha}} + |f_d(\tau)|)$$

Then defining $x(\tau, \xi)$ via (3.8) and $x_d(\tau)$ as the unique fixed point of (3.9), we have

$$\|x\|_{L_{d\rho}^{2,\alpha+\frac{1}{2};N-2}} \lesssim \|f\|_{L_{d\rho}^{2,\alpha;N}}$$

In fact, for the discrete spectral part we can improve this to

$$|x_d(\tau)| \lesssim \tau^{-N} \|f\|_{L_{d\rho}^{2,\alpha;N}}$$

The proof is a straightforward consequence of the fact that

$$|\xi^{-\frac{1}{2}} \sin \left[\lambda(\tau) \xi^{\frac{1}{2}} \int_{\tau}^{\sigma} \lambda^{-1}(u) du \right]| \lesssim \tau$$

What we shall effectively prove now is the following

Proposition 3.2. *Given N sufficiently large, $0 < \nu \leq \frac{1}{2}$, there is k sufficiently large and there exists a function $\tilde{e}_{2k-1} \in H^{\frac{\nu}{2}-}$ such that $\tilde{e}_{2k-1}|_{r \leq t} = e_{2k-1}|_{r \leq t}$ and such that (3.4) admits a fixed point $x(\tau, \cdot) \in \tau^{-N} L_{d\rho}^{2, \frac{1+\frac{\nu}{2}-}{2}}$, $\mathcal{D}_\tau x(\tau, \cdot) \in \tau^{-N-1} L_{d\rho}^{2, \frac{\nu}{2}-}$.*

Proof of Proposition 2.3, assuming Proposition 3.2. This is now a consequence of the fact that \mathcal{F} is an isometry from $H_{dR}^{2,\alpha}$ to $L_{d\rho}^{2,\alpha}$. The discrepancy between 2α on the one hand, and α on the other hand is due to the frequency being $\xi^{\frac{1}{2}}$ rather than ξ . \square

Proof of Proposition 3.2. We shall solve (3.4) via an iterative scheme, namely

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau) \mathcal{D}_\tau + \underline{\xi}) \underline{x}_j(\tau, \xi) = \mathcal{R}(\tau, \underline{x}_{j-1}) + f_{j-1}(\tau, \underline{\xi}), \quad (3.11)$$

with

$$\mathcal{R}(\tau, \underline{x}_{j-1}) = -4\beta_\nu(\tau) \mathcal{K} \mathcal{D}_\tau \underline{x}_{j-1} - \beta_\nu^2(\tau) \left(\mathcal{K}^2 + [\mathcal{A}, \mathcal{K}] + \mathcal{K} + \frac{\beta'_\nu}{\beta_\nu^2} \mathcal{K} \right) \underline{x}_{j-1} \quad (3.12)$$

and

$$f_{j-1}(\tau, \xi) = \mathcal{F} \left(\lambda^{-2}(\tau) [5(u_{2k-1}^4 - u_0^4) \tilde{\varepsilon}_{j-1} + RN(u_{2k-1}, \tilde{\varepsilon}_{j-1}) + R\tilde{e}_{2k-1}] \right) (\xi) \quad (3.13)$$

and we of course set

$$\tilde{\varepsilon}_j(\tau, R) = x_{d,j}(\tau) \phi_d(R) + \int_0^\infty x_j(\tau, \xi) \phi(R, \xi) \rho(\xi) d\xi, \quad j \geq 1,$$

while we also set $\tilde{\varepsilon}_0 = \underline{x}_0 = 0$. In particular, $f_0 = \mathcal{F}(R\tilde{\varepsilon}_{2k-1})$, $\mathcal{R}(\tau, \underline{x}_0) = 0$. This underlines the importance of the *first iterate* $\tilde{\varepsilon}_1$ since it will determine the smoothness of subsequent iterates due to the smoothing properties of the parametrix. We next turn to a careful analysis of the first iterate.

4. THE FIRST ITERATE

In light of the fact that $t^2 e_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{(\lambda t)^{2k}} IS^0(R^{1+}, Q')$ and the precise definition of this space from before, it is clear that e_{2k-1} is C^∞ -smooth except at the light cone $r = t$. Furthermore, by subtracting functions of regularity at least H_{dR}^1 and choosing k large enough, we may replace e_{2k-1} by an expression of the form³

$$t^2 e_{2k-1} = C(\tau) \tau^{-N-2} (\log(1-a))^i (1-a)^{\frac{v-1}{2}}, \quad C(\tau) = O(1), \quad (4.1)$$

for a finite collection of indices i . More precisely, we can write

$$t^2 e_{2k-1} = \sum_i C_i(\tau) \tau^{-N-2} (\log(1-a))^i (1-a)^{\frac{v-1}{2}} + E_{2k-1}$$

with $E_{2k-1} \in \tau^{-N} H_{R^2 dR}^1$.

The principal issue now becomes what the effect of the parametrix (3.8) on this kind of expression is. Then the key to proving Proposition 3.2 is the following result.

Lemma 4.1. *Define \tilde{e}_{2k-1} to be the function obtained from*

$$C \tau^{-N-2} (\log(1-a))^i (1-a)^{\frac{v-1}{2}}|_{r \leq t}$$

by truncation to $r \leq t$. Then the function $x(\tau, \xi)$ defined via (3.8) with

$$f(\tau, \xi) = \lambda(\tau)^{-2} \mathcal{F}(R\tilde{e}_{2k-1}(\tau, \cdot))(\xi)$$

satisfies $x = x^{(1)} + x^{(2)}$, where

$$x^{(1)}(\tau, \cdot) \in \tau^{-N} L_{d\rho}^{2, \frac{3}{4}-}, \quad x^{(2)} \in \tau^{-N} L_{d\rho}^{2, \frac{1+\frac{v}{2}}{2}}, \quad \frac{1}{R} \chi_{[R < \frac{v}{2}]} \mathcal{F}^{-1}(x^{(2)}) \in \tau^{-N} L_{dR}^\infty$$

provided k is sufficiently large. Finally, we also get the bound

$$\mathcal{D}_\tau x \in \tau^{-N-1} L_{d\rho}^{2, \frac{v}{2}-}, \quad |x_d(\tau)| \lesssim \tau^{-N-1}$$

where x_d is defined via (3.9), with f replaced by $R\tilde{e}_{2k-1}$.

Corollary 4.2. *Modify e_{2k-1} by restricting the terms*

$$C_i(\tau) \tau^{-N-2} (\log(1-a))^i (1-a)^{\frac{v-1}{2}}$$

to $r < t$ while smoothly truncating E_{2k-1} to a dilate of the light cone. Then the same bounds as in the preceding lemma apply to \underline{x} , the true first iterate.

³Recall that we may modify e_{2k-1} arbitrarily outside the light cone.

Proof of Corollary 4.2. This follows from Lemma 4.1 as well as Lemma 3.1 applied to E_{2k-1} . \square

Proof of Lemma 4.1. The proof of the bounds for small ξ proceeds exactly as in [8], and so we shall now focus on $\xi \gg 1$. The expression $\mathcal{F}(R\tilde{e}_{2k-1}(\tau, \cdot))(\xi)$ is given by

$$\mathcal{F}(R\tilde{e}_{2k-1}(\tau, \cdot))(\xi) = \int_0^\infty \phi(R, \xi) R\tilde{e}_{2k-1}(\tau, R) dR$$

and here we can restrict the integration to $0 \leq R \leq \nu\tau$. We shall next use the precise asymptotic expansion for $\phi(R, \xi)$. Specifically, for $R\xi^{\frac{1}{2}} \gtrsim 1$, we get

$$\phi(R, \xi) = a(\xi)f_+(R, \xi) + \overline{a(\xi)}f_-(R, \xi), \quad f_-(R, \xi) = \overline{f_+(R, \xi)} \quad (4.2)$$

where we have the expansions in Hankel functions

$$f_+(R, \xi) = e^{iR\xi^{\frac{1}{2}}} \sigma(R\xi^{\frac{1}{2}}, \xi),$$

$$\sigma(q, \xi) = \sum_{j=0}^{\infty} q^{-j} \psi_j^+(R)$$

and ψ_j^+ uniformly bounded in R with $\psi_j^+(R) = O(R^j)$. In particular, we get

$$f_+(R, \xi) = e^{iR\xi^{\frac{1}{2}}} + O(\xi^{-\frac{1}{2}})$$

for large ξ and $R\xi^{\frac{1}{2}} \gtrsim 1$.

Another key ingredient is the precise asymptotic formula for the coefficients $a(\xi)$. In fact, from [2], Cor. 3.7, we have

$$a(\xi) = \frac{\xi^{-\frac{1}{2}}}{2i} (1 + O(\xi^{-\frac{1}{2}})), \quad \xi \gg 1$$

Neglecting for now all terms of the form $O(\xi^{-\frac{1}{2}})$, we are then lead to the following integral:

$$\begin{aligned} & \mathcal{F}(R\tilde{e}_{2k-1}(\sigma, \cdot))(\xi) \\ &= \frac{\xi^{-\frac{1}{2}}}{2i} \int_0^{\nu\sigma} e^{iR\xi^{\frac{1}{2}}} R\tilde{e}_{2k-1}(\sigma, R) dR - \frac{\xi^{-\frac{1}{2}}}{2i} \int_0^{\nu\sigma} e^{-iR\xi^{\frac{1}{2}}} R\tilde{e}_{2k-1}(\sigma, R) dR \\ &= \frac{e^{i\nu\sigma\xi^{\frac{1}{2}}} \xi^{-\frac{1}{2}}}{2i} \int_0^{\nu\sigma} e^{i(R-\nu\sigma)\xi^{\frac{1}{2}}} R\tilde{e}_{2k-1}(\sigma, R) dR \\ &\quad - \frac{e^{-i\nu\sigma\xi^{\frac{1}{2}}} \xi^{-\frac{1}{2}}}{2i} \int_0^{\nu\sigma} e^{-i(R-\nu\sigma)\xi^{\frac{1}{2}}} R\tilde{e}_{2k-1}(\sigma, R) dR \end{aligned}$$

Recalling the definition of \tilde{e}_{2k-1} and replacing the outer factor R by $\nu\sigma$ up to an error of type E_{2k-1} , the preceding is seen to be equal to a linear

combination of terms of the form

$$\begin{aligned}
& (\log \sigma)^{N_2} \sigma^{-N_1} \cos(\nu \sigma \xi^{\frac{1}{2}}) \xi^{-\frac{1}{2}} \int_0^{\nu \sigma} \frac{(\log x)^i \sin(\xi^{\frac{1}{2}} x)}{x^{\frac{1-\nu}{2}}} dx \\
&= (\log \sigma)^{N_2} \sigma^{-N_1} \xi^{-\frac{3}{4+}} \cos(\nu \sigma \xi^{\frac{1}{2}}) C(\xi) \\
&+ \sum_{0 \leq j \leq i} (\log \sigma)^{N_2+j} \sigma^{-N_1-\frac{1-\nu}{2}} \xi^{-1-} \cos^2(\nu \sigma \xi^{\frac{1}{2}}) C_j(\xi) + O(\sigma^{-N_1-\frac{3-\nu}{2}} \xi^{-\frac{5}{4+}}), \\
&N_1 \geq N+1, \\
& (\log \sigma)^{N_2} \sigma^{-N_1} \sin(\nu \sigma \xi^{\frac{1}{2}}) \xi^{-\frac{1}{2}} \int_0^{\nu \sigma} \frac{(\log x)^i \cos(\xi^{\frac{1}{2}} x)}{x^{\frac{1-\nu}{2}}} dx \\
&= (\log \sigma)^{N_2} \sigma^{-N_1} \xi^{-\frac{3}{4+}} \sin(\nu \sigma \xi^{\frac{1}{2}}) C(\xi) \\
&+ \sum_{0 \leq j \leq i} (\log \sigma)^{N_2+j} \sigma^{-N_1-\frac{1-\nu}{2}} \xi^{-1-} \sin^2(\nu \sigma \xi^{\frac{1}{2}}) C_j(\xi) + O(\sigma^{-N_1-\frac{3-\nu}{2}} \xi^{-\frac{5}{4+}}), \\
&N_1 \geq N+1,
\end{aligned}$$

for some uniformly bounded functions $C(\xi), C_j(\xi)$ with symbol behavior. Note that the errors $O(\sigma^{-N_1-\frac{3-\nu}{2}} \xi^{-\frac{5}{4+}})$ are in $\sigma^{-N-2} L_{d\rho}^{2, \frac{1}{2}}$, which corresponds to $H_{R^2 dR}^1$ as for the error terms E_{2k-1} introduced further above. Now apply the parametrix (3.8) to the first two terms in the above two expansions, multiplied by $\lambda(\sigma)^{-2}$. Observe that

$$\lambda(\tau) \xi^{\frac{1}{2}} \int_{\tau}^{\sigma} \lambda^{-1}(u) du = \nu \tau (1 - (\frac{\tau}{\sigma})^{\frac{1}{\nu}}) \xi^{\frac{1}{2}}$$

and so we obtain the respective contributions

$$\begin{aligned}
x(\tau, \xi) &= \xi^{-\frac{5}{4+}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi\right) \frac{\rho^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi\right)}{\rho^{\frac{1}{2}}(\xi)} \sin\left[\nu \tau (1 - (\frac{\tau}{\sigma})^{\frac{1}{\nu}}) \xi^{\frac{1}{2}}\right] \\
&(\log \sigma)^{N_2} \sigma^{-N_1} \cos^{\kappa} \left(\nu \frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}} \xi^{\frac{1}{2}}\right) d\sigma, \quad \kappa = 1, 2,
\end{aligned}$$

as well as

$$\begin{aligned}
x(\tau, \xi) &= \xi^{-\frac{5}{4+}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi\right) \frac{\rho^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)} \xi\right)}{\rho^{\frac{1}{2}}(\xi)} \sin\left[\nu \tau (1 - (\frac{\tau}{\sigma})^{\frac{1}{\nu}}) \xi^{\frac{1}{2}}\right] \\
&(\log \sigma)^{N_2} \sigma^{-N_1} \sin^{\kappa} \left(\nu \frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}} \xi^{\frac{1}{2}}\right) d\sigma, \quad \kappa = 1, 2,
\end{aligned}$$

where now $N_1 \geq N + 1 + 2(\frac{\nu+1}{\nu})$. Expanding the sine function in the integrand, we encounter in the case $\kappa = 1$ the problematic terms

$$\begin{aligned} & \sin(\nu\tau\xi^{\frac{1}{2}})\xi^{-\frac{5}{4+}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{(\log \sigma)^{N_2}}{\sigma^{N_1}} \cos^2\left(\nu\frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}}\xi^{\frac{1}{2}}\right) d\sigma \\ & \cos(\nu\tau\xi^{\frac{1}{2}})\xi^{-\frac{5}{4+}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{(\log \sigma)^{N_2}}{\sigma^{N_1}} \sin^2\left(\nu\frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}}\xi^{\frac{1}{2}}\right) d\sigma \end{aligned}$$

where the bad⁴ terms $\cos^2(\nu\frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}}\xi^{\frac{1}{2}})$, $\sin^2(\nu\frac{\tau^{1+\nu^{-1}}}{\sigma^{\nu^{-1}}}\xi^{\frac{1}{2}})$ may be replaced up to an oscillating term by $\frac{1}{2}$, in which case the preceding further simplifies to

$$x^{(2)}(\tau, \xi) = \sin(\nu\tau\xi^{\frac{1}{2}})\xi^{-\frac{5}{4+}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} \frac{(\log \sigma)^{N_2}}{\sigma^{N_1}} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} d\sigma \quad (4.3)$$

as well as a similar term with $\cos(\nu\tau\xi^{\frac{1}{2}})$ in front. Here we have not gained the necessary decay for the Fourier coefficient x . On the other hand, this term has a well-defined oscillatory behavior in terms of $\xi^{\frac{1}{2}}$. Thus for $R < \frac{\nu\tau}{2}$, we get

$$\begin{aligned} \int_0^{\infty} \phi(R, \xi) x^{(2)}(\tau, \xi) \rho(\xi) d\xi &= \int_0^{\infty} \chi_{[R\xi^{\frac{1}{2}} \leq 1]} \phi(R, \xi) x^{(2)}(\tau, \xi) \rho(\xi) d\xi \\ &+ \int_0^{\infty} \chi_{[R\xi^{\frac{1}{2}} \geq 1]} \phi(R, \xi) x^{(2)}(\tau, \xi) \rho(\xi) d\xi \end{aligned}$$

In the first integral we have the implicit phase functions $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$, and performing integration by parts with respect to $\xi^{\frac{1}{2}}$ leads to a gain of $\simeq \tau^{-1}\xi^{-\frac{1}{2}}$, which makes the integrand absolutely integrable with respect to $d\xi$. In the second integral one has the implicit phases $e^{i(\pm R \pm \nu\tau)\xi^{\frac{1}{2}}}$, and here we gain $\simeq (\pm R \pm \nu\tau)^{-1}\xi^{-\frac{1}{2}}$. More precisely, for the first integral one writes (see [8])

$$\phi(R, \xi) = \phi_0(R) + R^{-1} \sum_{j=1}^{\infty} (R^2\xi)^j \phi_j(R^2), \quad (4.4)$$

with $\phi_j(R^2) \lesssim R^2$, and if this expression is hit by a $\partial_{\xi^{\frac{1}{2}}}$, it changes to

$$2R\xi^{\frac{1}{2}} \sum_{j=1}^{\infty} j(R^2\xi)^{j-1} \phi_j(R^2),$$

⁴in the sense of non-oscillatory

and we have $R^2 \lesssim R\xi^{-\frac{1}{2}}$ on the support of the integrand. The extra factor R is used to absorb the R^{-1} in

$$\left\| \frac{1}{R} \chi_{[R < \frac{\nu}{2}]} \mathcal{F}^{-1}(x^{(2)}) \right\|_{L_{dR}^\infty} \lesssim \tau^{-N_1+}$$

The case when the operator $\partial_{\xi^{\frac{1}{2}}}$ hits the other terms in the integral are handled similarly.

We have now reduced matters to controlling integrals of the form⁵

$$\begin{aligned} & \frac{\sin(\nu\tau\xi^{\frac{1}{2}})}{\xi^{\frac{5}{4+}}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) \frac{\rho^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho^{\frac{1}{2}}(\xi)} \Xi\left(\kappa\nu\frac{\tau^{1+\nu-1}}{\sigma^{\nu-1}}\xi^{\frac{1}{2}}\right) \frac{(\log\sigma)^{N_2}}{\sigma^{N_1}} d\sigma \\ & \frac{\cos(\nu\tau\xi^{\frac{1}{2}})}{\xi^{\frac{5}{4+}}} \int_{\tau}^{\infty} \frac{\lambda^{0+}(\tau)}{\lambda^{0+}(\sigma)} C\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) \frac{\rho^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho^{\frac{1}{2}}(\xi)} \Xi\left(\kappa\nu\frac{\tau^{1+\nu-1}}{\sigma^{\nu-1}}\xi^{\frac{1}{2}}\right) \frac{(\log\sigma)^{N_2}}{\sigma^{N_1}} d\sigma \end{aligned}$$

with $\Xi = \sin, \cos$ and $\kappa \in \{1, 2, 3\}$. Here we perform integration by parts with respect to σ , which gives the desired gain $\xi^{-\frac{1}{2}}$, placing the term into $L_{d\rho}^{2,1-}$.

To conclude the bounds for the first iterate, we still need to bound the contribution from all the errors which arose when we replaced $\phi(R, \xi)$ by $c \operatorname{Im}(i\xi^{-\frac{1}{2}} e^{iR\xi^{\frac{1}{2}}})$, as well as the errors of type E_{2k-1} . Note that the former type of error contributes to the Fourier transform of $R\tilde{e}_{2k-1}$ a term of the schematic form

$$\xi^{-1} \lambda(\sigma)^{-2} \int_0^{\nu\sigma} R\tilde{e}_{2k-1} dR$$

and we can use a crude L_{dR}^1 -bound for the integrand to infer that this contribution is bounded by $\lesssim \tau^{-N} \xi^{-\frac{3}{2}}$, which places this contribution into $\tau^{-N} L_{d\rho}^{2, \frac{3}{4}-}$. The contribution of errors of type E_{2k-1} was handled in the proof of Corollary 4.2. This completes the proof of the lemma. \square

5. THE SECOND ITERATE

We observe that Lemma 4.1 implies the following: the first iterate $\tilde{\varepsilon}_1$ can be written as $\tilde{\varepsilon}_1 = \tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}$ where

$$\begin{aligned} & \left\| \tilde{\varepsilon}^{(1)} \right\|_{H_{dR}^{\frac{3}{2}-}} + \left\| \frac{\tilde{\varepsilon}^{(1)}}{R} \right\|_{L_{dR}^M} \lesssim \tau^{-N}, \\ & \left\| \tilde{\varepsilon}^{(2)} \right\|_{H^{1+\frac{\nu}{2}-}} + \left\| \chi_{[R < \frac{\nu}{2}]} \frac{\tilde{\varepsilon}^{(2)}}{R} \right\|_{L_{dR}^\infty} \lesssim \tau^{-N} \end{aligned} \tag{5.1}$$

⁵These arise upon replacing $\sin^2(\nu\frac{\tau^{1+\nu-1}}{\sigma^{\nu-1}}\xi^{\frac{1}{2}})$ by $\sin^2(\nu\frac{\tau^{1+\nu-1}}{\sigma^{\nu-1}}\xi^{\frac{1}{2}}) - \frac{1}{2}$ and similarly for $\cos^2(\nu\frac{\tau^{1+\nu-1}}{\sigma^{\nu-1}}\xi^{\frac{1}{2}})$.

where $M \geq 2$ can be chosen arbitrarily large (with implicit constant depending on M). By radially, we then obtain

$$\left\| \frac{\tilde{\varepsilon}^{(2)}}{R} \right\|_{L_{dR}^\infty} + \left\| \frac{\tilde{\varepsilon}^{(2)}}{R} \right\|_{L_{dR}^M} \lesssim \tau^{-N},$$

To control the second iterate $\tilde{\varepsilon}_2$, write

$$((\mathcal{D}_\tau^2 + \beta_v(\tau)\mathcal{D}_\tau + \underline{\xi})(x_2 - x_1))(\tau, \underline{\xi}) = \mathcal{R}(\tau, \underline{x}_1) + \Delta_1 f_1(\tau, \underline{\xi})$$

with

$$\Delta_1 f_1(\tau, \underline{\xi}) := \mathcal{F}(\lambda^{-2}(\tau)[5(u_{2k-1}^4 - u_0^4)\tilde{\varepsilon}_1 + RN(u_{2k-1}, \tilde{\varepsilon}_1)])(\underline{\xi})$$

Then we claim

Lemma 5.1. *We have the estimates*

$$(x_2 - x_1)(\tau, \cdot) \in \tau^{-N} L_{d\rho}^{2,1}, | (x_2 - x_1)_d(\tau) | \lesssim \tau^{-N-1},$$

$$\mathcal{D}_\tau(x_2 - x_1)(\tau, \cdot) \in \tau^{-N-1} L_{d\rho}^{2, \frac{1}{2}}, |\partial_\tau(x_2 - x_1)_d(\tau)| \lesssim \tau^{-N-1}$$

hold. In fact, one gains a factor $\frac{1}{N}$ in the corresponding norm bounds.

Proof of Lemma 5.1. From Lemma 3.1, we conclude that, with x, f as in (3.8), we have

$$\begin{aligned} \sup_{\tau > 0} \tau^N \|x(\tau, \cdot)\|_{L^{2, \alpha + \frac{1}{2}}} &\lesssim \sup_{\tau > 0} \tau^{N+2} \|f(\tau, \cdot)\|_{L^{2, \alpha}} \\ \sup_{\tau > 0} \tau^N \|\mathcal{D}_\tau x(\tau, \cdot)\|_{L^{2, \alpha}} &\lesssim \sup_{\tau > 0} \tau^{N+1} \|f(\tau, \cdot)\|_{L^{2, \alpha}} \end{aligned}$$

Further, from [8], [2] we have the operator bounds (here α is arbitrary)

$$\begin{aligned} \mathcal{K}_{cc} : L_{d\rho}^{2, \alpha} &\rightarrow L_{d\rho}^{2, \alpha + \frac{1}{2}}, \quad [\mathcal{K}_{cc}, \mathcal{A}] : L_{d\rho}^{2, \alpha} \rightarrow L_{d\rho}^{2, \alpha + \frac{1}{2}} \\ \mathcal{K}_{dc} : L_{d\rho}^{2, \alpha} &\rightarrow \mathbb{R}, \quad \mathcal{K}_{dc} : \mathbb{R} \rightarrow L_{d\rho}^{2, \alpha}, \quad \mathcal{K}_{dd} : \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

In fact, the operator \mathcal{K}_{cc} is a smoothing operator, and corresponds to the operator \mathcal{K}_0 in [8]. Our strategy is to exploit the smoothing effect of the wave parametrix (3.8). This indeed leads to a derivative gain for all the terms in $\mathcal{R}(\tau, \underline{x}_1)$: assuming that the Fourier transform \underline{x}_1 of the first iterate can be decomposed into two terms with bounds as in Lemma 4.1, we get

$$\mathcal{R}(\tau, \underline{x}_1) \in \tau^{-N-2} L_{d\rho}^{2, \frac{1}{2}}$$

Application of (3.8) leads to expressions in $\tau^{-N} L_{d\rho}^{2,1}$; moreover, applying

\mathcal{D}_τ to these terms leads to expressions in $\tau^{-N-1} L_{d\rho}^{2, \frac{1}{2}}$, as required. The contribution to the discrete spectral part is also immediate.

We next turn to the contribution of $\Delta_1 f_1(\tau, \underline{\xi})$. In fact, we claim that all these

terms lead to a contribution in $L_{d\rho}^{2,1}$ upon application of (3.8). To see this, it suffices to check that

$$\mathcal{F}(\lambda^{-2}(\tau)[5(u_{2k-1}^4 - u_0^4)\tilde{\varepsilon}_1 + RN(u_{2k-1}, \tilde{\varepsilon}_1)])(\xi) \in \tau^{-N-2}L_{d\rho}^{2,\frac{1}{2}}$$

For the first term, use the relation from [8] that

$$u_{2k-1} - u_0 = O\left(\frac{\lambda^{\frac{1}{2}}R^{1+}}{(\lambda t)^2}\right)$$

from which we easily infer

$$\|\mathcal{F}(\lambda^{-2}(\tau)[5(u_{2k-1}^4 - u_0^4)\tilde{\varepsilon}_1])\|_{L_{d\rho}^{2,\frac{1}{2}}} \lesssim (\lambda t)^{-2}\|\tilde{\varepsilon}_1\|_{H_{dR}^1} \simeq \tau^{-2}\|\tilde{\varepsilon}_1\|_{H_{dR}^1}$$

which is bounded from Lemma 4.1. To control the source term $RN(u_{2k-1}, \tilde{\varepsilon}_1)$, we consider the two extreme possibilities

$$u_{2k-1}^3 \frac{\tilde{\varepsilon}_1^2}{R}, \quad \left(\frac{\tilde{\varepsilon}_1}{R}\right)^4 \tilde{\varepsilon}_1.$$

To bound the first term on the left, we use

$$\begin{aligned} \|\lambda^{-2}(\tau)u_{2k-1}^3 \frac{\tilde{\varepsilon}_1^2}{R}\|_{H_{dR}^1} &\lesssim \left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^M}^2 \|\lambda^{-2}(\tau)u_{2k-1}^3\|_{L^{2+}} \\ &\quad + \|\tilde{\varepsilon}_1\|_{H_{dR}^{1+}} \left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^M} \|\lambda^{-2}(\tau)u_{2k-1}^3\|_{W^{1,\infty}} \end{aligned}$$

while for the second term we have

$$\left\|\left(\frac{\tilde{\varepsilon}_1}{R}\right)^4 \tilde{\varepsilon}_1\right\|_{H_{dR}^1} \lesssim \left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^{10}}^5 + \left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^M}^4 \|\tilde{\varepsilon}_1\|_{H_{dR}^{1+}}$$

and we have

$$\left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^{10}} \lesssim \left\|\frac{\tilde{\varepsilon}_1}{R}\right\|_{L_{dR}^M} + \|\tilde{\varepsilon}_1\|_{H_{dR}^1}$$

Note that the rapid decay rate of $\tilde{\varepsilon}_1$ gives much more than τ^{-N-2} -decay.

6. THE HIGHER ITERATES

Here we repeat the procedure of the preceding section, except that we replace $\underline{x}_2 - \underline{x}_1$ by $\underline{x}_k - \underline{x}_{k-1}$, $k \geq 3$, and we replace one copy of $\tilde{\varepsilon}_1$ by $\tilde{\varepsilon}_{k-1} - \tilde{\varepsilon}_{k-2}$ in each source term. Then we can literally repeat what we did before. To be specific, we claim the following:

Lemma 6.1. *We have the bounds*

$$(x_k - x_{k-1})(\tau, \cdot) \in \tau^{-N}L_{d\rho}^{2,1}, \quad |(x_k - x_{k-1})_d(\tau)| \lesssim \tau^{-N-1},$$

$$\mathcal{D}_\tau(x_k - x_{k-1})(\tau, \cdot) \in \tau^{-N-1}L_{d\rho}^{2,\frac{1}{2}}, \quad |\partial_\tau(x_k - x_{k-1})_d(\tau)| \lesssim \tau^{-N-1}$$

In fact, one gains a factor $(\frac{1}{N})^k$ in the corresponding norm bounds.

The proof is by induction on k , and is in all respects identical to the one of the preceding lemma, except that there is no one factor $\tilde{\varepsilon}_{k-1} - \tilde{\varepsilon}_{k-2}$ involved in the analogue of $\Delta_1 f_1(\tau, \underline{\xi})$. The factor $(\frac{1}{N})^k$ comes from the repeated time integrations. \square

The fixed point of (3.4) is now found by iteration, which completes the proof of Proposition 2.3. \square

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JOACHIM KRIEGER

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland
joachim.krieger@epfl.ch

WILHELM SCHLAG

Department of Mathematics, The University of Chicago
5734 South University Avenue, Chicago, IL 60615, U.S.A.
schlag@math.uchicago.edu